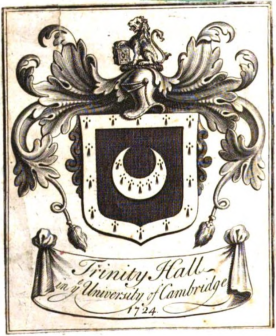
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TRACTS

ON THE

Resolution of Affected Algebräick Equations

BY

Various Methods of Approximation.

TRACTS

ON THE

Resolution of Affected Algebräick Equations

BY

DR. HALLEY'S, MR. RAPHSON'S,

AND

SIR ISAAC NEWTON'S,

METHODS OF APPROXIMATION.

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THE PREFACE.

THE principal object of the present Collection of Tracts is to explain, and illustrate by examples, Mr. Raphson's and Dr. Halley's Methods of Resolving Affected Algebräick Equations by Approximation, and to compare these two methods with each other, in order to be able to form a judgement of their respective merits and determine to which of them we ought to give the preference. With this view I have, in the beginning of the Book, reprinted Dr. Halley's Discourse on this subject from the Second Volume of the Collection of Tracts called *Miscellanea Curiosa*, which was published in the year 1708. In this Discourse (which is written in a very concise and obscure style,) Dr. Halley, first, gives an account of what had been done by Mr. *Raphson* and Monsieur *De Lagny* of Paris with respect to this business of resolving equations by approximation, and particularly in the resolution of *pure* equations, or the extraction of the roots of numbers, and then proceeds to describe his own method of resolving *affected* equations of all degrees

by

by approximation, which he confiders as a great improvement of the method that had been given by Mr. Raphfon for the fame purpofe. This defcription is but fhort, being contained in only two pages, to wit, pages 12 and 13 of the prefent publication; and it is, in my opinion, very obfcure and difficult to underftand: but it is illuftrated in the following pages of the Tract, (to wit, pages 14, 15, &c - - - 21,) by being applied to the refolution of three numeral equations, to wit, the cubick equation $x^3 - 17x^2 + 54x = 350$, the incompleat biquadratick equation $x^4 * - 3x^2 + 75x = 10,000$, and the compleat biquadratick equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. This Difcourfe of Dr. Halley I have endeavoured to explain and illuftrate in a very long Appendix to it, which extends from page 25 to page 183, and of which the principal contents are as follows:

The firft 12 pages of this Appendix, to wit, pages 25, 26, &c, - - - - 37, contain a very full defcription of the grounds of both Dr. Halley's and Mr. Raphfon's methods of refolving equations by approximation, and, indeed, of all other methods of the fame kind; which confift of the four following operations; to wit, in the firft place, to find, (by a few eafy conjectures and trials, with fmall numbers confifting of one, or two, figures, or by fome other method fuggefted by the conditions of the equation, or of the Problem from which it is derived,) a tolerably near value of x, or the unknown root of the propofed equation; and, fecondly, (calling the faid near value a,) to put z for the unknown difference between a and the root x; and, if a is lefs than x, and confequently

consequently x is equal to $a + z$, to substitute the binomial quantity $a + z$ instead of x in the terms of the original equation; and, if a is greater than x, and consequently x is equal to $a - z$, to substitute $a - z$ instead of x in the terms of the original equation; by which substitution the original equation, of which the unknown quantity x was the root, will be transformed into another equation, of which the unknown quantity z (which is much less than x, and usually less than a tenth part, and often less than a hundredth part, of it,) will be the root; and, thirdly, to omit, or reject from the said second, or transformed, equation, of which z is the root, all the terms that involve any higher powers of z than it's square zz, or than it's simple power z; and then, in the 4th and last place, to resolve the remaining equation (which will be either a quadratick equation or a simple equation, according as the terms involving zz are retained or rejected,) in the common methods given for that purpose. For by these operations we shall obtain a near value of the second unknown quantity z, and consequently a near value of $a + z$, or $a - z$, or of the root x of the original equation, which will be nearer to it's true value than the former near value of it, a, was. These operations are the grounds of both Dr. Halley's and Mr. Raphson's methods of approximation: and the difference between them is only this; to wit, that Dr. Halley retains, in the second, or transformed, equation, of which z is the root, all the terms that involve the square of z as well as those that involve it's simple power, and is thereby under the necessity of resolving a quadratick equation in order to obtain the value of z; whereas

Mr. Raphſon rejects all the terms that involve the ſquare of z as well as thoſe that involve z^3, z^4, z^5, and it's other higher powers, and retains only the terms that involve the ſimple power of z, or z itſelf, and by ſo doing is enabled to find the value of z by reſolving only a ſimple equation. The conſequence of this difference is, that the value of z obtained by a ſingle proceſs of Dr. Halley's method will be more exact than the value of it obtained by a ſingle proceſs of Mr. Raphſon's method; but the difficulty of obtaining it is conſiderably greater in the former method than in the latter. How far one of theſe advantages counter-balances the other, and which of the two methods, upon the whole, deſerves to be preferred to the other, can only be determined by a cloſe and carefull compariſon of the two methods with each other in the application of them to the reſolution of the very ſame numeral equations, and by carrying the inveſtigations of the roots of thoſe equations to the ſame, or nearly the ſame, degree of exactneſs, or to the ſame, or nearly the ſame, number of figures, by both the methods. This therefore is what I have endeavoured to do, with reſpect to the three numeral equations above-mentioned, (which have been reſolved by Dr. Halley in the preceeding Tract,) in the ſubſequent pages of this Appendix.

The cubick equation $x^3 - 17x^2 + 54x = 350$ is that which is firſt examined: and the reſolution of it begins in page 37, and ends in page 49, taking up 12 pages. In pages 38 and 39 I ſhew how, by various eaſy reaſonings, conjectures and trials, we may conclude that the true value of x is ſomewhat leſs than the number 15.

And then, putting a for 15, and e (in conformity to Dr. Halley's notation,) for the unknown difference of a and x, I substitute $a - e$ instead of x in the equation $x^3 - 17x^2 + 54x = 350$, and thereby obtain the transformed equation

$$\left\{ \begin{array}{l} \quad a^3 - 3a^2e + 3ae^2 - e^3 \\ - 17a^2 + 34ae - 17e^2 \\ + 54a - 54e \end{array} \right\}$$

$= 350$, which, (by substituting in it, instead of a, a^2, and a^3, their several numeral values 15, 225, and 3375, and by making the subtractions and additions of the terms which are necessary to bring all the unknown terms, or terms involving e, to the left-hand side of the equation,) becomes $219e - 28e^2 + e^3 = 10$. From this equation I then reject e^3, as being very small in comparison of $219e$ and $28e^2$, and thereby obtain the quadratick equation $219e - 28e^2 = 10$; which, by a division of all it's terms by 28, is reduced to the equation $\frac{219e}{28} - e^2 = \frac{10}{28}$. And, this equation being resolved by the proper methods of resolving quadratick equations, we find that it's two roots are $\frac{219}{56} + \frac{216.427,817 \ \&c}{56}$, and $\frac{219}{56} - \frac{216.427,817, \ \&c}{56}$; of which, (by proper reasonings grounded on our knowledge of the limits between which the original unknown quantity x must lie,) we conclude that the lesser root $\frac{219}{56} - \frac{216.427,817, \ \&c}{56}$ must be the value of e that is wanted for our purpose. And therefore

we conclude that e, or the difference between 15 and x, will be nearly $= \frac{219}{56} - \frac{216.427,817, \&c}{56}$ ($= \frac{2.572,183, \&c}{56}$) $= 0.045,932$, and consequently that x, or $a - e$, or $15 - e$, will be, nearly, $= 15 - 0.045,932 = 14.954,068$. Q. E. I.

Having thus gone through the resolution of the equation $x^3 - 17x^2 + 54x = 350$ by Dr. Halley's method, I proceed to resolve it by Mr. Raphson's method, beginning the approximation to the value of x from the same number 15, which was taken for a, or it's first near value, in the foregoing resolution of the equation by Dr. Halley's method.

By substituting $a - e$ instead of x in the equation $x^3 - 17x^2 + 54x = 350$ we transform it, as before, into the equation

$$\left\{\begin{array}{l} \quad\quad a^3 - 3a^2e + 3ae^2 - e^3 \\ - 17a^2 + 34ae - 17e^2 \\ + 54a - 54e \end{array}\right\}$$

$= 350$; and, by rejecting from this equation the three terms $3ae^2 - 17e^2 - e^3$, (which involve the square and cube of e,) we reduce it to the simple equation $a^3 - 17a^2 + 54a - 3a^2a + 34ae - 54e = 350$, which (by substituting in it, instead of a, a^2, and a^3, their respective values 15, 225, and 3375, and by making the proper subtractions and additions of the terms, to bring all the unknown terms, or terms involving e, to the left-hand side

ſide of the equation,) becomes $219e = 10$. And, this eaſy ſimple equation being reſolved (by dividing both ſides of it by 219, the co-efficient of e,) we have $e = \frac{10}{219} = 0.0456$, and conſequently x, or $a - e$, or $15 - e$, nearly $= 15 - 0.0456 = 14.9544$. Q. E. I.

This proceſs is evidently much ſhorter and ſimpler than the foregoing proceſs by Dr. Halley's method, and gives us the value of x, or the root of the original equation $x^3 - 17x^2 + 54x = 350$, exact to five places of figures, the error of the number 14.9544 beginning only in the laſt figure 4, which in the more accurate value of x is a cypher. This, however, is a very conſiderable degree of exactneſs obtained by a ſingle proceſs of Mr. Raphſon's method, which is much eaſier than the former, or correſponding, proceſs of Dr. Halley's method.

I then proceed, in pages 46, 47, 48, and 49, to find a more exact value of x, or the root of the propoſed equation $x^3 - 17x^2 + 54x = 350$, by means of a ſecond proceſs of Mr. Raphſon's method, taking 14.954, (or the five figures that are exact in the number 14.9544, which was obtained by means of the former proceſs,) for the ground-work of the new proceſs. I therefore ſubſtitute 14.954 inſtead of x in the trinomial quantity $x^3 - 17x^2 + 54x$, and finding that the reſult of this ſubſtitution is $= 349.985,150,664$, which is leſs than the abſolute term 350, I conclude that 14.954 is leſs than the true value of x, and, putting c for 14.954, and f for the unknown difference by which this number falls

ſhort

ſhort of the true value of x, I ſubſtitute the binomial quantity $c + f$ inſtead of x in the equation $x^3 - 17x^2 + 54x = 350$; by which it is transformed into the equation

$$\left\{\begin{array}{rrrr} c^3 & + 3c^2f & + 3cf^2 & + f^3 \\ -17c^2 & - 34cf & - 17f^2 & \\ +54c & + 54f & & \end{array}\right\} = 350.$$

I then reject from this equation the three terms $3cf^2$, $-17f^2$, and $+f^3$, agreeably to Mr. Raphſon's directions; and the remaining terms form the ſimple equation $c^3 + 3c^2f - 17c^2 - 34cf + 54c + 54f$ nearly $= 350$, which (by ſubſtituting in it's terms, inſtead of c, c^2, and c^3, their ſeveral numeral values, and by making ſuch additions and ſubtractions of the ſeveral terms as are neceſſary to bring all the terms that involve the unknown quantity f to the left-hand ſide of the equation,) produces the ſimple equation $216.430,348 \times f = 0.014,849,336$. And the reſolution of this laſt equation, (by dividing both ſides of it by $216.430,348$, or the co-efficient of f,) gives us f $\left(= \frac{0.014,849,336}{216.430,348}\right) = 0.000,068,610,2$. Therefore x, or $c + f$, or $14.954 + f$, will be nearly $(= 14.954 + 0.000,068,610,2) = 14.954,068,610,2$. Q. E. I.

This number 14.954,068,610,2 is probably true in all it's twelve figures. But it is certainly ſo in the firſt ten figures 14.954,068,61; of which the firſt eight figures 14.954,068, are the ſame as thoſe of the former value of

of x, which was obtained by Dr. Halley's method of approximation. So that one process of Dr. Halley's method of approximation gives us the value of x in the cubick equation $x^3 - 17x^2 + 54x = 350$ exact to eight figures by the resolution of the quadratick equation $219e - 28e^2 = 10$, which has two roots, of which the lesser is the value of e wanted for our purpose; and two processes of Mr. Raphson's method of approximation give us the value of the same quantity exact to at least ten places of figures by the resolution of the two simple equations $219e = 10$, and $216.430,348 \times f = 0.014,849,336$. The reader must now judge for himself to which of these two methods he will give the preference. For my own part, I give it to Mr. Raphson's method.

The next equation to be considered is the incompleat biquadratick equation $x^4 * - 3x^2 + 75x = 10,000$. This I therefore proceed next to resolve, first, by Dr. Halley's method and afterwards by Mr. Raphson's, in a very full and distinct manner, in pages 49, 50, 51, &c -- 65.

And, in the first place, in pages 50 and 51 I find, by means of some easy conjectures and trials, that the value of x in this equation will be greater than 9, but less than 10, and nearer to 10 than to 9. I therefore make $a = 10$, and, putting e for the difference between a and x, substitute $a - e$ instead of x in the proposed equation $x^4 * - 3x^2 + 75x = 10,000$; which is thereby transformed into the equation

a^4

$$\left\{\begin{array}{l} a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ - 3a^2 + 6ae - 3e^2 \\ + 75a - 75e \end{array}\right\}$$

$= 10,000$. From this equation I then reject, according to Dr. Halley's directions, the two terms $- 4ae^3 + e^4$, which involve in them the cube and fourth power of e, and thereby reduce the said equation to the quadratick equation

$$\left\{\begin{array}{l} a^4 - 4a^3e + 6a^2e^2 \\ - 3a^2 + 6ae - 3e^2 \\ + 75a - 75e \end{array}\right\} = 10,000,$$

which (by substituting in it, instead of a, a^2, a^3, and a^4, their several values 10, 100, 1000, and 10,000, and making the several additions and subtractions of the terms which are necessary to bring all the terms that involve the unknown quantity e to the left-hand side of the equation,) becomes changed into the numeral equation $4015e - 597e^2 = 450$, and (by dividing all the terms by 597, or the co-efficient of e^2,) into the equation $6.725,293 \times e - e^2 = 0.753,769$, which is a quadratick equation duely prepared for resolution.

This quadratick equation I then proceed to resolve, and find that it has two roots, to wit, $3.362,646 + 3.248,633$ and $3.362,646 - 3.248,633$, of which (by considering the limits of the magnitude of x, or $a - e$, which is known to be less than 10, but greater than 9, which makes it necessary for e to be less than 1,) I find that

that the lesser root 3.362,646 — 3.248,633 must be that which will be suited to our purpose, or will be nearly equal to the difference between a, or 10, and x. I therefore conclude that e will be nearly $= 3.362,646 - 3.248,633 = 0.114,013$, and therefore that x, or $a - e$, or $10 - e$, will be nearly $= 10 - 0.114,013$ or (neglecting the three last figures 013, as probably not exact) $= 10 - 0.114 = 9.886$. Q. E. I.

This number 9.886 appears, upon a trial of it, to be very nearly equal to, but somewhat less than, the true value of x in the proposed equation $x^4 * - 3x^2 + 75x = 10,000$. In order therefore to find a more accurate value of x, I put c for 9.886, and f for it's unknown difference from the true value of x, and substitute the binomial quantity $c + f$ instead of x in the equation $x^4 * - 3x^2 + 75x = 10,000$; by means of which substitution the said equation is transformed into the equation

$$\left\{\begin{array}{l} \quad c^4 + 4c^3f + 6c^2f^2 + 4cf^3 + f^4 \\ - 3cc - 6cf - 3f^2 \\ + 75c + 75f \end{array}\right\}$$

$= 10,000$, of which the small quantity f is the root. I then reject from this equation the two terms $4cf^3$ and f^4, which reduces it to the quadratick equation

$$\left\{\begin{array}{l} \quad c^4 + 4c^3f + 6c^2f^2 \\ - 3cc - 6cf - 3f^2 \\ + 75c + 75f \end{array}\right\} = 10,000;$$

and (by substituting in this last equation, instead of c, c^2, c^3, and c^4, their several values 9.886 and it's square, cube,

cube, and fourth power, and by making the feveral additions and fubtractions of the terms which are neceffary to bring all the terms that involve the unknown difference f to the left-hand fide of the equation,) this equation is converted into the numeral equation 3880.437, 593,824 $\times f$ + 583.397,976 $\times ff$ = 0 010,480,863,984; and this equation (by dividing all it's terms by 583.397, 976, or the numeral co-efficient of ff) is further reduced to the equation 6.651,441,646 $\times f + ff$ = 0.000,017,965,204,569,033, which is a quadratick equation duely prepared for refolution. This equation (which has only one root,) is then refolved, but not without a great deal of laborious calculation; and it's root f is found to be = 0.000,002,700. Therefore x, or $c + f$, or 9.886 + f, is nearly (= 9.886 + 0.000,002,700) = 9.886,002,700; that is, the root of the propofed biquadratick equation is nearly equal to 9.886,002,700.

Q. E. I.

Dr. Halley makes this root to be nearly equal to 9.886,260,393,649,5. But I believe he muft have made fome miftake in his calculation, becaufe the other value of x here found, to wit, 9.886,002,700, is confirmed by the fecond refolution which I have made of this equation by Mr. Raphfon's method. This fecond refolution is as follows:

Let 10 be taken for a, or the firft near value of x in the equation $x^4 * - 3x^2 + 75x = 10,000$, as it was in the foregoing refolution of that equation by Dr. Halley's method; and let $a - e$ be fubftituted inftead of x in

in the three terms x^4, $3x^2$, and $75x$. Then will the transformed equation resulting from such substitution be (as before)

$$\left\{\begin{array}{l} \quad a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ - 3a^2 + 6ae - 3e^2 \\ + 75a - 75e \end{array}\right\}$$

$= 10,000$; which, by rejecting the four terms $6a^2e^2 - 3e^2 - 4ae^3 + e^4$ according to the directions of Mr. Raphson, will be reduced to the simple equation

$$\left\{\begin{array}{l} \quad a^4 - 4a^3e \\ - 3a^2 + 6ae \\ + 75a - 75e \end{array}\right\} = 10,000.$$

And this last equation will (by substituting in it's terms, instead of a, a^2, a^3, and a^4, their several values 10, 100, 1000, and 10,000, and by making such additions and subtractions of the terms as are necessary in order to bring all the terms that involve the unknown quantity e to the left-hand side of the equation,) be converted into the short numeral equation $4015e = 450$. And the resolution of this last equation (by dividing both sides of it by 4015, the co-efficient of the unknown quantity e,) gives us e $(= \frac{450}{4015}) = 0.112$. Therefore x, or $a - e$, or $10 - e$, will be nearly $= 10 - 0.112 = 9.888$; or the second near value of x in the proposed equation $x^4 * - 3x^2 + 75x = 10,000$, obtained by this first process of Mr. Raphson's method of approximation, will be 9.888. Q. E. I.

The

The operations of this firſt proceſs of Mr. Raphſon's method of approximation are, it is evident, much ſhorter and eaſier than thoſe of the firſt proceſs of Dr. Halley's method, which required the reſolution of the quadratick equation $6.725,293 \times e - e^2 = 0.753,769$.

I then proceed to find a third near value of the root of the propoſed equation $x^4 * - 3x^2 + 75x = 10,000$ by means of a ſecond proceſs of Mr. Raphſon's method of approximation. And for this purpoſe I, in the firſt place, ſubſtitute the laſt near value of x juſt now obtained, to wit, the number 9.888, in the terms of the trinomial quantity $x^4 * - 3x^2 + 75x$, in order to diſcover whether the value of the ſaid trinomial quantity thence reſulting will be greater or leſs than 10,000, or the abſolute term of the propoſed equation. And I find, upon making this ſubſtitution, that the value of the ſaid trinomial quantity thence reſulting is 10,007.752,728, 231,936; which is a little greater than 10,000, or the abſolute term of the propoſed equation. I therefore conclude that 9.888 is a little greater than the true value of x in the equation $x^4 * - 3x^2 + 75x = 10,000$, and, putting f for the unknown difference between 9.888 and x, I ſubſtitute the binomial quantity $9.888 - f$ inſtead of x in the propoſed equation $x^4 * - 3x^2 + 75x = 10,000$, by which (omitting the terms that involve either f^2, f^3, or f^4, and ranging the terms in their proper order,) we arrive at the ſimple equation $3882.771,660,288 \times f = 7.752,728,231,936$; the reſolution of which gives us $f \left(= \frac{7.752,728,231,936}{3882.771,660,288}\right) = 0.001,99$.

Therefore

Therefore x, or $9.888 - f$, will be nearly $= 9.888 - 0.001,99 = 9.886,01$; or the third near value of x in the proposed equation $x^4 * - 3x^2 + 75x = 10,000$, obtained by this second process of Mr. Raphson's method of approximation, will be 9.886,01. Q. E. I.

Here again we may observe that the labour of resolving the simple equation $3882.771,660,288 \times f = 7.752,728,231,936$ in this second process of Mr. Raphson's method is much less than the labour of resolving the quadratick equation $6.651,441,646 \times f + ff = 0.000,017,965,204,569,033$ in the second process of the foregoing resolution of the proposed equation by Dr. Halley's method.

I then proceed to find a fourth near value of x in the proposed equation $x^4 * - 3x^2 + 75x = 10,000$, by means of a third process of Mr. Raphson's method of approximation: and, for that purpose, I, first, substitute the last near value of it, to wit, 9.886,01, instead of x, in the trinomial quantity $x^4 * - 3x^2 + 75x$, in order to discover whether the value of the said trinomial quantity resulting from such substitution is greater, or less, than 10,000, or the absolute term of the proposed equation. And I find, upon making this substitution, that the said result is $= 10,000.028,313,570,294,077,144,01$; which is a little greater than the said absolute term. I therefore conclude that 9.886,01 is a little greater than the true value of x, and, putting g for the unknown difference between 9.886,01 and x, I substitute the binomial quantity $9.886,01 - g$ instead of x in the proposed

posed equation $x^4 * - 3x^2 + 75x = 10,000$, but with an omission of the terms that involve either g^2, g^3, or g^4, agreeably to Mr. Raphson's directions; and by this substitution I transform the said equation into a simple equation, which, when the terms of it are properly ranged, becomes the equation $3880.449,261,795,383,204 \times g = 0.028,313,570,294,077,144,01$. And, this last equation being resolved by dividing both sides of it by the co-efficient of g, we shall have g ($=$

$$\frac{0.028,313,570,294,077,144,01}{3880.449,261,795,383,204}) = 0.000,007,296,467,$$

and consequently x, or $9.886,01 - g$, $= 9.886,01 - 0.000,007,296,467 = 9.886,002,703,533$; that is, the fourth near value of x in the proposed equation $x^4 * - 3x^2 + 75x = 10,000$, obtained by this third process of Mr. Raphson's method of approximation, will be 9.886, 002,703,533. Q. E. I.

Of this number 9.886,002,703,533 it is almost certain that (if no mistakes have been made in the calculation,) the first ten figures 9.886,002,703 are exact. And, as the first nine of them, to wit, 9.886,002,70, are the same with the first nine figures of the last value of x that had been found by the second process of Dr. Halley's method of approximation, to wit, 9.886,002,700, I think we may be confident that, at least, these nine figures must be exact, and therefore that Dr. Halley's number 9.886, 260,393,649,5, given above in page 15 for a most accurate value of x scarce exceeding the truth by 2 in the last figure, must be erroneous.

It

It ſeems to me, from the compariſon of the two methods of approximation in this example, as well as in the laſt, that Mr. Raphſon's method (which proceeds by the reſolution of only ſimple equations) is ſo much ſimpler and eaſier, both to underſtand and to practiſe, than Dr. Halley's method (which proceeds by the reſolution of quadratick equations,) that, notwithſtanding the advantage of the latter method in giving us more new figures of the root ſought exactly, in every ſingle proceſs of it, than are given by a ſingle proceſs of the former method, it very much deſerves to be preferred to the latter, or Dr. Halley's, method.

I next proceed to Dr. Halley's third example, which is the biquadratick equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. This equation he juſtly conſiders as a very difficult one, becauſe it is of that form which admits of four different affirmative roots, and becauſe the co-efficients of x and x^2, to wit, 14,937 and 1998, are very great numbers in compariſon of the reſolvend, or abſolute term, 5000. And in truth this equation has four affirmative roots, to wit, the decimal fraction 0.350,986,045,866,06 &c, the mixt number 12.756,441,794,480,744,022, &c, the mixt number 32.060,290, &c, and the mixt number 34.832,280, &c. But of theſe roots Dr. Halley finds only the ſecond, or leaſt but one, to wit, 12.756,441,794,480,744,022, &c. This root therefore I, in page 65, proceed to inveſtigate, firſt, by Dr. Halley's method of approximation, and afterwards by Mr. Raphſon's.

In the first place I conjecture that x is nearly equal to 10, and I substitute 10 instead of x in the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$, and I find the value of the said quantity resulting from that substitution to be 19,570, which is almost four times as great as 5000, or the absolute term of the proposed equation. I then form a second conjecture about it's magnitude, and suppose it to be, nearly, $= 12$, and substitute 12 instead of x in the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$, and find the result of this substitution to be 9,036. This result, though much less than the former result 19,570, is yet much greater than the absolute term 5000.

By these two conjectures and trials I find that, while x increases from 10 to 12, the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ decreases from 19,570 to 9,036. I therefore suppose that, if x were to increase further from 12 to 13, the said quadrinomial quantity would decrease further from 9,036 to some lesser number. And so, upon trial, I find it to do. For, if x is $= 13$, the said quadrinomial quantity will be $= 3,718$, which is less than the absolute term 5000. I therefore now conclude with certainty that x must be greater than 12, but less than 13.

Further, in order to obtain a still nearer value of x, I make use of the following conjectural, but very probable, and (as appears upon trial,) very useful, supposition. Since, when x is $= 12$, the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 9,036$, and, when

when x is equal to the root of the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, the said quadrinomial quantity is $= 5000$, and, when x is $= 13$, the said quadrinomial quantity is $= 3,718$, it seems probable that the difference of the first and third of the first three quantities 12, x, and 13, to wit, the difference $13 - 12$, or 1, will be to the difference of the first and second of these quantities, to wit, $x - 12$, in, nearly, the same proportion as the difference of the first and third of the latter three quantities 9,036, 5000, and 3,718, which correspond to the first and third quantities, 12 and 13, of the first set, is to the difference of the first and second of the latter three quantities 9,036, 5000, and 3,718, which correspond to the first and second quantities, 12 and x, of the first set; that is, that 1 will be to $x - 12$ in, nearly, the same proportion as $9,036 - 3,718$, or 5,318, is to $9,036 - 5000$, or 4,036, and consequently that $x - 12$ will be nearly $= \frac{1 \times 4,036}{5,318} = \frac{4,036}{5,318} = 0.7$ &c, and therefore that x will be nearly $= 12.7$.

Having thus obtained 12.7 for a first near value of x that is sufficiently near to it's true value to become the basis, or ground-work, of a farther approximation to it's true value by either Dr. Halley's, or Mr. Raphson's method of approximation, I proceed to make use of it for obtaining a more exact value of x by Dr. Halley's method of approximation. And for that purpose I begin by substituting it instead of x in the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to discover

whether

whether the value of the said quadrinomial quantity resulting from such substitution will be greater or less than 5000, or the absolute term of the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Now the result of this substitution is 5298.6559. And therefore it appears that, while x increases from 12.7 to 13, the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will decrease from 5298.6559 to 3,718. And therefore, when x is of some intermediate value between 12.7 and 13, the said quadrinomial quantity will be equal to the absolute term 5000, or, in other words, the true value of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be greater than 12.7, but less than 13. We must therefore now make $12.7 = a$, and $12.7 + e$, or $a + e$, $= x$, and substitute $a + e$ instead of x in the proposed equation, but with an omission of the terms that involve either e^3 or e^4. This substitution will produce the transformed equation

$$\left\{\begin{array}{llll} & 14,937a & + 14,937e & \\ - & 1998a^2 & - 3996ae & - 1998e^2 \\ + & 80a^3 & + 240a^2e & + 240ae^2 \\ - & a^4 & - 4a^3e & - 6a^2e^2 \end{array}\right\}$$

=, nearly, 5000. And (by substituting in this equation, instead of a, a^2, a^3, and a^4, their several values, to wit, 12.7, and it's square, cube, and fourth power, and making the several additions and subtractions of the terms which are necessary to bring all the terms that involve

involve the unknown quantity e to the left-hand ſide of the equation, with only a known number on it's right-hand ſide,) we further reduce it to the equation $5296.132 \times e - 82.26 \times e^2 = 298.6559$, and (by dividing all the terms by 82.26, or the co-efficient of e^2,) to the equation $64.382,834,9 \times e - e^2 = 3.630,633,357,646,48$, which is a quadratick equation properly prepared for reſolution.

This quadratick equation (which has two roots,) is then reſolved, but not without a great deal of laborious calculation: and it's two roots are found to be $32.191,417,4 + 32.134,976,5$, and $32.191,417,4 - 32.134,976,5$, of which it is evident that the former root (which is greater than 64) cannot be the value of e that is ſuited to the preſent purpoſe, or is nearly equal to the difference between 12.7 and the true value of x, which we know to be leſs than 13; and therefore we conclude that the ſecond root, $32.191,417,4 - 32.134,976,5$, is the root that is to be adopted on this occaſion. We thereſore conclude that e, or the difference between x and 12.7 will be nearly $= 32.191,417,4 - 32.134,976,5$, or 0.056,440,9, and conſequently that x, or $a + e$, or $12.7 + e$, will be, nearly, $(= 12.7 + 0.056,440,9) = 12.756,440,9$. O. E. I.

This value of x is exact in the firſt ſeven figures 12.756,44, it's more accurate value, as computed both by Dr. Wallis and Dr. Halley, being 12.756,441,794,480,744,02.

Dr. Halley on this occasion points out a correction to be made to the value of x just now obtained, to wit, the number 12.756,440,9, without entering upon a compleat second process of his method of approximation, and tells us that we may, by this correction, find the value of x to be = 12.756,441,794,48, or to thirteen places of figures, all exact. This correction I have endeavoured to explain and to put in practice, in pages 73, 74, and 75. But I am not sure that I have perfectly understood it; and the value of x resulting from my application of it does not agree with the value assigned by Dr. Halley, to wit, 12.756,441,794,48, but differs from it in some of the latter figures, being 12.756,441,794,387. The reason of this difference I do not know. But, as I do not approve of this correction for the reasons alledged in art. 34, pages 76 and 77, I shall say no more of it in this place, but shall proceed to give an account of the compleat second process of Dr. Halley's method of approximation which I have gone through in pages 77, 78, 79, - - - 82, and by which I find the more accurate value of x to be 12.756,441,794,480,744,022,60; of which I believe the first 20 figures, to wit, 12.756,441, 794,480,744,022, to be exact.

In order to begin the said second process, I substitute 12.756,44 instead of x in the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$; and I find the value of the said quantity resulting from such substitution to be 5000.009,486,644,489,476,503,04; which is a very little greater than 5000, or the absolute term of the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$.

5000. And I thence conclude, for the reasons given in art. 29, that the number 12.756,44 is somewhat less than the true value of x.

I therefore put c for the number 12.756,44, and f for it's unknown difference from the true value of x, and substitute the binomial quantity $c + f$ instead of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$; by which substitution the said equation is transformed into the equation

$$\left\{\begin{array}{llll} & 14,937c + 14,937f & & \\ - & 1998c^2 - 3996cf - 1998ff & & \\ + & 80c^3 + 240c^2f + 240cff + 80f^3 & & \\ - & c^4 - 4c^3f - 6c^2ff - 4cf^3 - f^4 & & \end{array}\right\}$$

$= 10,000$, or (omitting the three terms $80f^3$, $4cf^3$ and f^4) into the equation

$$\left\{\begin{array}{ll} & 14,937c + 14,937f \\ - & 1998c^2 - 3996cf - 1998ff \\ + & 80c^3 + 240c^2f + 240cff \\ - & c^4 - 4c^3f - 6c^2ff \end{array}\right\}$$

=, nearly, 10,000. And, if we substitute in this last equation, instead of c, c^2, c^3, and c^4, their several values, to wit, 12.756,44, and it's square, cube, and fourth power, and make the several additions and subtractions that are necessary to bring all the terms that involve the unknown quantity f to the left-hand side of the equation, this last equation will be converted into the equa-

tion

tion 5286.568,162,865,159,936 × f — 87.184,431,158,4 × ff = 0.009,486,644,489,476,503,04, and finally (by dividing all the terms of the laſt equation by 87.184,431,158,4, the co-efficient of ff,) into the equation 60.636,607,851,008,642,270,08 × f — ff = 0.000,108,811,221,951,323,000,352,097,689,829,824,386,0 &c, which is a quadratick equation duely prepared for reſolution.

This equation is then reſolved in page 82, but not without a great deal of laborious calculation; and it's leſſer root (which is that which is equal to the difference between 12.756,44 and x) is found to be 0.000,001,794,480,744,022,60. Therefore x, or $c + f$, or 12.756,44 + f, will be, nearly, (= 12.756,44 + 0.000,001,794,480,744,022,60) = 12.756,441,794,480,744,022,60; that is, the value of x in the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be very nearly equal to 12.756,441,794,480,744,022,60. Q. E. I.

I then proceed in pages 83, 84, 85, &c - - - - 96, to find the value of the ſame root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ by Mr. Raphſon's method of approximation.

In making this approximation I begin with the ſame firſt near value of x with which I began the former approximation by Dr. Halley's method, to wit, the number 12.7, putting a = 12.7, and $x = a + e$ = 12.7 + e, and ſubſtituting the binomial quantity 12.7 + e inſtead of

of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, but with an omission of all the terms that involve either e^2, e^3, or e^4. This substitution produces the simple equation $14,937a - 1998a^2 + 80a^3 - a^4 + 14,937e - 3996ae + 240a^2e - 4a^3e =$, nearly, 5000. And this last equation, being further reduced (by substituting in it's terms instead of a, a^2, a^3, and a^4 their several numeral values, to wit, 12.7, and it's square, cube, and fourth power, and by making the several additions and subtractions which are necessary to bring all the terms involving e to the left-hand side of the equation, and to consolidate them into one term) becomes $5296.132 \times e = 298.6559$; the resolution of which gives e $\left(= \frac{298.6559}{5296.132}\right) =$ 0.056,39. Therefore x, or $a + e$, or $12.7 + e$, will be $(= 12.7 + 0.056,39) = 12.756,39$; that is, the second near value of x, obtained by this first process of Mr. Raphson's approximation, will be 12.756,39. Q. E. I.

Of this number 12.756,39 the first five figures 12.756 are exact, the more accurate value of x being 12.756, 441,794,480, &c.

I then substitute 12.756,39 instead of x in the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$, and find the result of the said substitution to be = 5000.273,805,270,591,690,609,59; which is somewhat greater than 5000, or the absolute term of the proposed equation. Therefore, for the reasons before given, I conclude that 12.756,39 must be less than the true value of x. I therefore put $c = 12.756,39$, and f for the unknown quantity by which 12.756,39 falls short of the

true

true value of x, and substitute $c+f$ instead of x in the proposed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, but with an omission of all the terms that involve either f^2, f^3, f^4. And the transformed equation thence arising is

$$\left\{\begin{array}{rrrl} & 14{,}937c & + & 14{,}937f \\ - & 1998c^2 & - & 3996cf \\ + & 80c^3 & + & 240c^2f \\ - & c^4 & - & 4c^3f \end{array}\right\}$$

nearly, $= 5000$; in which if we substitute for c, c^2, c^3, and c^4, their several numeral values, to wit, 12.756,39, and it's square, cube, and fourth power, and afterwards make the several additions and subtractions of the terms which are necessary to consolidate the four terms involving the unknown quantity f into one term, and to consolidate all the known quantities in the equation into one term, and bring it to the opposite side of the equation, we shall obtain the simple equation $5286.576{,}881{,}150{,}968{,}476 \times f = 0.273{,}805{,}270{,}591{,}690{,}609{,}59$, by the resolution of which we shall have f ($= \dfrac{0.273{,}805{,}270{,}591{,}690{,}609{,}59}{5286.576{,}881{,}150{,}968{,}476}$) $= 0.000{,}051{,}792$. Therefore $c+f$, or $12.756{,}39 + f$, will be ($= 12.756{,}39 + 0.000{,}051{,}792$) $= 12.756{,}441{,}792$; that is, the third near value of x, which is obtained by this second process of Mr. Raphson's method of approximation, will be 12.756,441,792. Q. E. I.

Of this number 12.756,441,792 the first ten figures, 12.756,441,79, are exact, the more accurate value of x

being

being 12.756,441,794,480,744,02, as has been obſerved before. But this greater degree of exactneſs may be attained by carrying this approximation by Mr. Raphſon's method one ſtep further; which may be done as follows:

If the number 12.756,441,792, obtained by the laſt proceſs, be ſubſtituted inſtead of x in the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$, the value of the ſaid quantity reſulting from ſuch ſubſtitution will be 5000.000,013,114,621,596,455,067,255,345,516,642, 304; which is a little greater than 5000, or the abſolute term of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore, for the reaſons before given, 12.756,441,792 muſt be ſomewhat leſs than the true value of x. I therefore put $d = 12.756,441,792$, and ſuppoſe x to be equal to $d + g$, or to 12.756,441, 792 + g, and ſubſtitute that binomial quantity, inſtead of x, in the propoſed equation, but with an omiſſion of all the terms that involve either g^2, g^3, or g^4. And the transformed equation thence obtained, after it's terms are properly arranged, becomes the ſimple equation 5,286.567,850,393,729,132,515,900,260,352 $\times g =$ 0.000,013,114,621,596,455,067,255,345,516,642,304, as is ſhewn in pages 93, 94, and 95. Therefore g is ($= \frac{0.000,013,114,621,596,455,067,255,345}{5286.567,850,393,729,132,515,900,262,352}$) = 0.000,000, 002,480,744,022,88; and conſequently x, or 12.756, 441,792 + g, will be = 12.756,441,792 + 0.000,000, 002,480,744,022,88 = 12.756,441,794,480,744,022,88; that is, the fourth near value of x, which is obtained by this

this third procefs of Mr. Raphfon's method of approximation, will be 12.756,441,794,480,744,022,88.

Q. E. D.

This value of x (which has been obtained by means of thefe three proceffes of Mr. Raphfon's method of approximation,) agrees with the value of it above-obtained by means of two proceffes of Dr. Halley's method of approximation, to wit, 12.756,441,794,480,744,022,60, in the firft twenty figures 12.756,441,794,480,744,022; and therefore thefe twenty figures are probably exact, and the true value of x is greater than 12.756,441,794, 480,744,022, but lefs than 12.756,441,794,480,744,023.

Both thefe methods of attaining the value of x in this equation to this great degree of exactnefs have been attended with a great deal of labour; but I have found the labour neceffary to the three proceffes of Mr. Raphfon's method confiderably lefs than the labour neceffary to the two proceffes of Dr. Halley's method. The readers, who fhall have gone carefully through all the operations of both the methods, will be able to judge for themfelves which of the two methods is the clearer and eafier, and deferves, upon the whole, to be preferred to the other.

The refolutions of the three foregoing equations $x^3 - 17x^2 + 54x = 350$, $x^4 * - 3x^2 + 75x = 10{,}000$, and $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, (which Dr. Halley had produced as examples of his method of approximation,) having been compleated in page 96, I proceed in page 97 to mention, and explain at large, another

another method of resolving high affected equations by approximation, which is totally different from both the former methods. This method, I think, may with propriety be called *The Differential Method of Approximation;* because it is grounded on a very useful property of the differences of the root x of any proposed equation and any two near values of x. This property may be thus described. Let the capital letter A be put for the absolute term of the equation, which is the value of the compound quantity which forms the left-hand side of the equation, when the true value of x is substituted instead of x in the said compound quantity. Let b be a near value of x, (the nearer the better,) either greater or less than it's near value; and let c be another near value of x, either greater or less than it's true value. And let the capital letter B denote the value of the compound quantity which forms the left-hand side of the equation, when b is substituted instead of x in the terms of the said compound quantity; and let the capital letter C denote the value of the said compound quantity when c is substituted in it's terms instead of x; so that the three quantities B, A, and C shall be the values of the said compound quantity corresponding to the three quantities b, x, and c, respectively, or resulting from the substitution of b, x, and c in the said compound quantity, respectively. These things being supposed, the property of the differences of x and it's two near values b and c is as follows. "The difference of b and "c, the first and third of the three former quantities "b, x, and c, will be to the difference of b and x, the "first and second of the said three quantities, in, nearly, "the

"the same proportion as the difference of B and C, the "first and third of the three latter quantities B, A, and "C, is to the difference of B and A, the first and second "of the said three latter quantities."

By means of this proportion the value of x may be derived to a considerable degree of exactness from the quantities b, c, B, A, and C, which are all known quantities.

It was by means of this proportion that, in seeking a, or the first near value of x in the last-mentioned biquadratick equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, in order to make it the basis of a process of Dr. Halley's method of approximation, when we had found that 12 was something less, and that 13 was something greater, than the true value of x in that equation, and that, if 12 was substituted instead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, the result would be $= 9{,}036$, and that, if 13 was substituted instead of x in the same compound quantity, the result would be $= 3{,}718$, we discovered that 12.7 would be much nearer to the true value of x than either 12 or 13. For in that case the three first quantities b, x, and c were 12, x, and 13, and the three latter quantities B, A, and C, corresponding to them, were 9,036, 5000, and 3,718, and we supposed $13 - 12$ to be to $x - 12$ in nearly the same proportion as $9{,}036 - 3{,}718$ is to $9{,}036 - 5000$, or 1 to be to $x - 12$ in nearly the same proportion as 5,318 is to 4,036; whence it followed that

that $x - 12$ would be nearly $= \frac{4,036}{5,318} = 0.7$ &c, and consequently that x would be nearly equal to 0.7 &c $+ 12$, or 12.7.

And, if we had carried the division of 4,036 by 5,318 to two figures more in the quotient, we should have found $x - 12$ to be nearly $= 0.758$, and consequently x to be nearly $= 12.758$, of which the four first figures 12.75 are exact, and the fifth figure 8 is but a little too great, the more accurate value being 12.756,441, &c. So that we might by this means have obtained the four figures 12.75, which are all exact, for the value of a, or of the first near value of x, which was to be made the basis, or ground-work, of a further approach to the true value of x by a process of Dr. Halley's method of approximation. This property therefore of the differences of x and it's two near values b and c appears from this example to be highly useful in obtaining a much nearer value of the root x than either of it's two former near values b and c, or, in this instance, 12 and 13, which we had obtained before.

This property of the differences of x and it's two near values b and c takes place with respect to three logarithms that are nearly equal to each other. For it is the ground of the method by which, when three numbers, which are nearly equal to each other, are given, and the logarithms of the greatest and least of them are known, the logarithm of the middle number is derived from the known logarithms of the two extreme numbers, and from the three numbers themselves. For in that case, if the

the three numbers be called B, A, and C, and the logarithms of B and C are known, but that of A is unknown, it is cuſtomary to make the following proportion, in order to diſcover the logarithm of A; to wit, log. B — log. C is to log. B — log. A :: B — C : B — A; whence it follows that log. B — log. A will be $= \overline{\text{log. B} - \text{log. C}} \times \dfrac{\overline{B - A}}{B - C}$, and (adding log. A to both ſides) log. B will be $= \overline{\text{log. B} - \text{log. C}} \times \dfrac{B - A}{B - C} + \text{log. A}$, and conſequently log. A will be $= \text{log. B} - \dfrac{B - A}{B - C} \times \overline{\text{log. B} - \text{log. C}}$. Indeed the known application of this proportion to the finding of the logarithms of ſuch intermediate numbers between other numbers whoſe logarithms are known, was the circumſtance that firſt ſuggeſted to me the idea of applying it to the finding of a nearer value of the root x of any propoſed equation when we are already poſſeſſed of two contiguous near values of it, as b and c, and of the reſults of the ſubſtitution of the ſaid near values in the compound quantity that forms the left-hand ſide of the equation.

This *Differential Method* of approximating to the roots of affected equations is explained pretty fully, and illuſtrated by an example, in the Scholium in pages 97, 98, 99, &c - - - 109. And it is there ſhewn that it gives us about as many new figures of the value of x exact as are obtained by one proceſs of Mr. Raphſon's method of

of approximation. However, I think, Mr. Raphson's method is, upon the whole, somewhat preferable to it, for the reasons assigned in art. 50, pages 105, 106, and 107; and I would therefore recommend this differential method to be used only in the first part of the investigation of x, or the root of any proposed equation, while we are endeavouring to find a, or the first near value of x, (which is to be made the basis of a nearer approach to the true value of it by Mr. Raphson's method of approximation,) to a considerable degree of exactness, as, for example, to three or four places of figures. For in this way, I believe, this differential method will be found extremely useful.

After the Scholium concerning the differential method of approximation (which ends in page 109,) I have inserted some observations of Mr. Raphson on Monsieur de Lagny's and Dr. Halley's claims to the merit of having invented their methods of resolving equations by approximation, which shew them to be only branches, or modifications, of his method, which he had considered with attention and deliberately rejected, as being less simple and convenient in practice than his own method. These observations are contained in pages 109 and 110.

In page 111 I enter upon an inquiry, whether it may not sometimes, in extracting the square-root of a number to many places of figures, (as was necessary in resolving the quadratick equations that occurred in the resolution of the three foregoing equations by Dr. Halley's method of approximation,) be convenient to proceed by some

method of approximation rather than by the common method of extracting the ſquare root. This inquiry extends to page 132; and the reſult is, "that, from all "the trials I have there made of different expreſſions for "approximating to the value of the ſquare-root of a given "number N, or $aa + b$, I am inclined to conclude that, "in performing the extractions of the ſquare-roots of "given numbers which are neceſſary in Dr. Halley's "method of reſolving high affected equations, it will "almoſt always be found eaſier and more convenient to "proceed by the common method of extracting them "than to have recourſe to either of the three expreſſions "found for that purpoſe in the courſe of this inquiry, "not excepting even Mr. Raphſon's ſimple expreſſion "$a + \frac{b}{2a}$, which is very much to be preferred to the "two others."

I then in pages 133, 134, &c reſume the conſideration of the three foregoing equations which Dr. Halley has adduced in his Tract as examples of his method of reſolving equations by approximation, and I inquire whether theſe equations may not have ſome other roots beſides thoſe that have been inveſtigated by Dr. Halley and in the Appendix to his Diſcourſe. And I ſhew, firſt, that the firſt of theſe equations, to wit, the cubick equation $x^3 - 17x^2 + 54x = 350$, cannot have any other root beſides that above inveſtigated, to wit, 14.954,068,61. This is ſhewn by two different methods, in pages 133, 134, 135, &c, - - 142. And I then ſhew that the biquadratick equation $x^4 * - 3x^2 + 75x = 10{,}000$ cannot

cannot have any other root besides that above investigated, to wit, 9.886,002,70. This is shewn in pages 142, 143, and 144. And I then proceed to examine the third and last of the said equations, to wit, the biquadratick equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and to inquire, first, whether this equation can have any other root greater than the root 12.756,441, 794, &c which we have already found, and secondly, whether it can have any other root less than the said root already found. The former of these inquiries begins in page 144, and ends in page 152; and it contains a clear and regular proof that this equation will have two roots that are greater than the root 12.756,441,794, &c, of which the lesser will be something less, and the greater will be something greater, than the number 33.42. The second inquiry begins in page 152, and ends in page 155; and it contains a proof that this equation will also have a fourth root, that will be less than the root 12.756,441, 794, &c already found. And, in pages 156, 157, 158, it is shewn that this fourth root will be nearly = to the decimal fraction 0.35; and, in pages 158, 159 and 160, it is further shewn by means of only one process of Mr. Raphson's method of approximation, that the first six figures of the value of this fourth root will be 0.350,987. A second process of that method of approximation would have given us the decimal fraction 0.350,987,045,856,06 for it's more accurate value. I then in pages 161, 162, &c, - - - - 175, investigate to a moderate degree of exactness the two greatest roots of this equation, which had been before found to be not very different from the number 33.42; and I find that the lesser of those roots

will be = 32.060,290, &c, and that the greater of them will be = 34.832,280, &c. And thus the resolutions of all the three equations $x^3 - 17x + 54 = 350$, $x^4 * - 3x^2 + 75x = 10,000$, and $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, which Dr. Halley had produced as examples of his method of resolving affected equations by approximation, are rendered quite compleat and satisfactory.

The remaining eight pages 176, 177, 178, &c,- - 183, of the Appendix to Dr. Halley's Tract above-mentioned, contain no Algebräick calculations, but are employed, first, in describing the circumstance that forms the difference between the accurate methods of resolving Algebräick equations and the methods of resolving them by approximation, and, secondly, in giving an account of the principal methods of resolving Algebräick equations by approximation that have been published, and of the times of their publication. The former of these points is treated of in pages 176, 177, and 178, and the latter in pages 178, 179, 180, &c - - 183.

The next Tract in this Collection, after this long Appendix to Dr. Halley's Discourse, contains a very full exhibition of Dr. Wallis's Solution of a very difficult Arithmetical Problem that had been proposed to him by Colonel *Silas Titus*, a person of great note in the reign of King Charles the Second, to whom he had at one time had the honour of being a gentleman of the bed-chamber. And Dr. Wallis further tells us that this Problem had been originally proposed to Colonel Titus (as

(as the Colonel informed him,) by Dr. *John Pell*, the famous Algebräist of that time. So that it is a question that has engaged the attention, and exercised the talents, of very eminent Mathematicians. But my reason for introducing it into this Collection of Tracts (which relates to the resolution of equations, and not to the solution of problems, or the reduction of them to equations,) was it's connection with the biquadratick equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, which has been the object of our attention and consideration through so great a part of the foregoing Appendix to Dr. Halley's Tract: for this circumstance, I supposed, would make my readers desirous of becoming acquainted with the Problem from which it had been derived, and of understanding the Solution that had been given of that Problem by Dr. Wallis. This, however, I found to be no easy matter: for I attempted several times to understand Dr. Wallis's Solution of this Problem without being able to make it out, though I had each time employed several hours in the attempt, and had filled some pages of large sheets of paper with the long Algebräick operations of multiplication, addition, and subtraction, which were necessary for that purpose. But these operations were so intricate and complicated that I repeatedly made mistakes in attempting to perform them, and, after much tiresome labour, was obliged to begin them over again. At last, however, I performed them rightly, and obtained the equation which has been given us by Dr. Wallis, as the result of the conditions of the Problem, and as being the grand, final, equation by the resolution of which the unknown quantities which the Problem requires us

to find, may be discovered. And this equation is an equation of the 12th order, or involves the 12th power of the unknown quantity which is it's root. And, as I should be sorry that my readers should be forced to waste their time and pains in making the same fruitless endeavours to perform these operations that I had made, I have, in this Tract, set down, at full length, the several compound quantities produced by every new operation in Dr. Wallis's Solution; so that the diligent reader, that shall chuse to go through the Solution with care and exactness, and to perform all the necessary operations in it, may continually compare the several terms of his own results with those of the results here set down, and thereby discover and correct any mistakes he may have made in performing the operations, as soon as they arise. The consideration of this Problem begins in page 187, and Dr. Wallis's Solution of it ends in page 217.

This Problem is, to find three numbers, denoted by the three letters a, b, and c, that shall be of such magnitudes that $aa + bc$ shall be $= 16$, and $bb + ac$ shall be $= 17$, and $cc + ab$ shall be $= 18$. And, if l be put for 16, m for 17, and n for 18, and ee be put $= 2aa$, the final equation involving e, resulting from the conditions of the Problem, is the equation of the 12th order, (or involving the 12th power of the unknown quantity e,) that is set down in page 217.

But this equation may be reduced to an equation of the eighth power, or order, by dividing it by the trinomial quantity $e^4 - 4le^2 + 4l^2$, or the square of $2l - e^2$; which

which division is set down by Dr. Wallis in his Algebra, and is likewise set down in the present Volume in pages 218 and 219. And the equation resulting from this division is set down in page 220. And, in pages 220 and 221, the values of the three letters l, m, and n, to wit, the numbers 16, 17, and 18, are substituted in the terms of the said equation instead of the said letters themselves, and the said equation is thereby converted into the numeral equation $e^8 - 80e^6 + 1998e^4 - 14,937e^2 + 5000 = 0$; and this equation, by proper additions and subtractions of it's terms, is further changed into the equation $14,937e^2 - 1998e^4 + 80e^6 - e^8 = 5000$; and this last equation, by substituting x instead of ee in it's terms, becomes the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, which was so much the object of our attention in the Appendix to Dr. Halley's Discourse.

In the equation of the 12th order, which is the result of Dr. Wallis's Solution of the foregoing Problem, and which is set down in page 217 of the present Volume, all the terms are placed on the left-hand side of the equation, and their result, or value, is declared to be equal to 0; which is the method in which Dr. Wallis and most other writers of Algebra, since the publication of Harriot's Treatise of Algebra in the year 1631, usually range the terms of their equations. And this equation is afterwards reduced to an equation of the eighth order by dividing it by the trinomial quantity $e^4 - 4le^2 + 4l^2$, which is an exact divisor of it, or which divides it so as to produce a quotient consisting of a certain number of terms, without any remainder, one of which terms is e^8;

and this quotient is also equal to o. Thus we have a dividend that is equal to o divided by the trinomial quantity $e^4 - 4le^2 + 4l^2$, and producing a quotient that is also equal to o. All this seems very obscure and mysterious, and bordering upon nonsense. And therefore, to avoid the difficulties arising from this method of reducing the said equation of the 12th order to an equation of the eighth order, I have, in art. 15, pages 222, 223, 224, &c, - - - - - 227, given another method of doing the same thing, which is perfectly clear and intelligible. For, by adding some quantities to both sides of Dr. Wallis's final equation, set down in page 217, of which the right-hand side was o, I obtain two finite compound quantities that are equal to each other; and I then divide these two equal quantities by the trinomial quantity $e^4 - 4le^2 + 4l^2$, and thereby obtain two separate quotients, which will be equal to each other. And from this equation between these two quotients we may afterwards derive (by a proper arrangement of it's terms, and by substituting in it, instead of the letters l, m, and n, the values of those letters, to wit, the numbers 16, 17, and 18,) first, the equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$, and, afterwards, (by substituting x instead of ee in this last equation) the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, which was resolved in the foregoing Appendix to Dr. Halley's Discourse. This article, I presume, will be thought an interesting one by all such readers as are desirous of preserving clear ideas in all the operations that they have occasion to perform in Algebra.

In

In pages 228, 229, 230, &c, - - 232, I have applied the number 12.756,441,794,480,744, or the second value of x in the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, or of ee in the equation $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$, to the Solution of Colonel Titus's Problem, by taking $aa = \frac{ee}{2}$ $(= \frac{12.756,441,794,480,744}{2}) = 6.378,220,897,240,372$, and consequently $a = \sqrt{}(6.378,220,897,240,372) = 2.525,513,986$, or, nearly, 2.525,514; which is therefore the value of the first of the three numbers that were required to be found by that Problem. This first number a being thus found, the second number b may be derived from it by computing the expression $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$, as is shewn in the course of Dr. Wallis's Solution of the Problem; and the third number c may be derived from the two former numbers a and b by computing the expression $\frac{l - aa}{b}$, as is likewise shewn in Dr. Wallis's Solution. These computations are made in pages 229 and 230, and it there appears that the expression $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$ is $= 2.969,152,826$, or, nearly, 2.969,153, and that the expression $\frac{l - aa}{b}$ is $= 3.240,580$. Therefore the three numbers sought are, nearly, the three mixt numbers 2.525,514, 2.969,153, and 3.240,580. Q. E. I.

And

And in art. 17 and 18, pages 230, 231, and 232, it is ſhewn that theſe three numbers will anſwer the conditions of the Problem.

But Dr. Wallis has obſerved that the firſt, or leaſt, value of ee in the equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$ will alſo enable us to find three other numbers that will anſwer the conditions of the Problem, as well as the three numbers 2.525,514, 2.969,153, and 3.240,580. And this is ſhewn in art. 19 of the preſent Tract, pages 232, 233, &c, - - 256. For this firſt, or leaſt, value of ee is $= 0.350{,}987{,}045{,}866$, &c as has been ſhewn in the foregoing Appendix. And, if we take $ee = 0.350{,}987{,}045{,}866$, &c, and $aa = \frac{ee}{2}$ $\left(= \frac{0.350{,}987{,}045{,}866,\ \&c}{2}\right) = 0.175{,}493{,}522{,}933$, &c, and conſequently a $(= \sqrt{0.175{,}493{,}522{,}933},\ \&c) = 0.418{,}919{,}470{,}701$, &c, and from this value of a deduce the values of b and c by computing the expreſſions $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$ and $\frac{l - aa}{b}$, we ſhall find the former of theſe expreſſions, or the value of the ſecond number b, to be $= 3.912{,}226{,}866$, and the latter of them, or the value of the third number c, to be $= 4.044{,}884{,}670$. And conſequently the three numbers ſought will be the decimal fraction 0.418,919,470, &c, and the two mixt numbers 3.912,226,866, and 4.044,884,670. This is ſhewn in art. 19, pages 232, 233, and 234; and in art. 20, pages 234, 235, and 236, it is ſhewn

ſhewn that theſe three numbers will anſwer the conditions of the Problem.

After thus diſcovering that two of the values of ee in the equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$ ſerve equally well to enable us to find three numbers that will anſwer the conditions of Colonel Titus's Problem, it may be ſuſpected that the other two values of ee in the ſame equation, to wit, 32.060,290,8 and 34.832,280,2, would likewiſe enable us to find two other ſets of numbers that would alſo anſwer the conditions of the Problem. But this is found, upon trial, not to be true. For, if ee is taken $=$ 32.060,290,8, and aa is taken $= \frac{ee}{2}$ $(= \frac{32.060,290,8}{2}) = 16.030,145,4$, and a is taken $= \sqrt{16.030,145,4}$, and b is taken $= \frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$, or to $\frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18{,}432}}{36}$, and c is taken $= \frac{l - aa}{b}$, or $\frac{16 - aa}{b}$, it will be found that the values of $aa + bc$, $bb + ac$, and $cc + ab$, reſulting from theſe values of a, b, and c, will not be equal to the numbers 16, 17, and 18 reſpectively; but that $aa - bc$ (inſtead of $aa + bc$) will be $= 16$, and that $bb + ac$ will be (as before) $= 17$, and that $cc - ab$ (inſtead of $cc + ab$) will be $= 18$; and conſequently that the three values of a, b, and c ſo obtained will be fitted to anſwer the conditions of a problem ſomewhat different from that of Colonel Titus, to

wit,

wit, a problem in which it ſhould be required to find the values of three unknown quantities a, b, and c, that ſhould be of ſuch magnitudes that $aa - bc$ (inſtead of $aa + bc$,) ſhould be equal to 16, and that $bb + ac$ ſhould (as before,) be equal to 17, and that $cc - ab$ (inſtead of $cc + ab$,) ſhould be equal to 18. And, if cc is taken $= 34.832,280,2$, and aa is taken $= \frac{cc}{2}$ $\left(= \frac{34.832,280,2}{2}\right) = 17.416,140,1$, and b is taken $= \frac{17a}{36} - \frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}$, and c is taken $= \frac{l - aa}{b}$, or $\frac{16 - aa}{b}$, it will be found that the values of $aa + bc$, $bb + ac$, and $cc + ab$, reſulting from theſe values of a, b, and c, will not be equal to the numbers 16, 17, and 18 reſpectively, but that $aa - bc$ (inſtead of $aa + bc$) will be $= 16$, and that $bc - ac$ (inſtead of $bc + ac$,) will be $= 17$, and that $cc + ab$ will be (as before) $= 18$; and conſequently that the three values of a, b, and c ſo obtained will be fitted to anſwer the conditions of a problem ſomewhat different both from Colonel Titus's Problem and from the other problem juſt now mentioned, to wit, a problem in which it ſhould be required to find the values of three unknown quantities a, b, and c, that ſhould be of ſuch magnitudes that $aa - bc$ (inſtead of $aa + bc$) ſhould be equal to 16, and that $bc - ac$ (inſtead of $bc + ac$) ſhould be equal to 17, and that $cc + ab$ ſhould (as before) be equal to 18. And thus it appears that the equation $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$ is related to, or may be derived from, three different problems that bear a reſemblance to each other

other by agreeing in some of their conditions, but not in all, and that two of the values of *ee* in this equation, to wit, the least and the least but one, are related to the first of the said three problems, to wit, that proposed by Colonel Titus, and will enable us to find two sets of numbers that will answer the conditions of that problem; and that the third value of *ee*, in this equation, to wit, 32.060,290,8, will enable us to find three numbers that will answer the conditions of the second problem; and that the fourth, or greatest, value of *ee* in the same equation, to wit, 34.832,280,2, will enable us to find three numbers that will answer the conditions of the third problem. This is a remarkable conclusion, and deserves to be attended to. These observations are briefly mentioned in art. 21, pages 236, 237, and 238; but, for more ample satisfaction concerning these two greatest values of *ee* in the said equation, the reader is referred to Dr. Wallis's Algebra, Chapter LXII, art. 63, 64, 65, 66, &c, - - 79, where he will find the subject fully discussed, but not, it is apprehended, without some degree of obscurity arising, partly, from the consideration of negative quantities, and, partly, from the doctrine of the generation of equations, one from another, by multiplication, or by bringing all the terms of the equation to the first, or left hand, side of the equation, so as to make them equal to 0, and then multiplying the equations (so prepared and made equal to nothing,) one into another, and from the eminently-false position derived from that manner of generating equations, to wit, "that every Algebräick equation has as many roots as there are units in the index of the highest power of the unknown quantity

tity contained in the equation." For this doctrine of the generation of equations, one from another, by multiplication, (which was invented by Harriot, and adopted by Des Cartes, and Dr. Wallis, and almost all the subsequent writers on Algebra,) instead of being an improvement in that science (as many people consider it), has, in my opinion, been of great detriment to it by destroying it's simplicity and perspicuity, and therefore ought again to be discarded from it.

The foregoing Solution of Colonel Titus's Problem, given us by Dr. Wallis, is (as we have already observed,) exceedingly tedious and laborious: and great part of the labour required in the said Solution arises from the necessity we are under, of raising the equation which involves the unknown quantity a to the 12th degree, or to the 12th power of a, in order to free it's terms from radicality. But Mr. William Frend, the ingenious author of the late perspicuous Treatise on Algebra in one volume, octavo, intitled, *Principles of Algebra*, (in which he totally rejects the absurd and perplexing doctrine of *negative quantities*, or *quantities less than nothing*, or *quantities obtained by the subtraction of a greater quantity from a lesser*,) has lately communicated to me another Solution of this Problem which produces only a biquadratick equation, to wit, the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. And this equation is found to have three roots, or, in the language of modern Algebräists, three *real and affirmative* roots; of which the middle root will enable us to find the first set of values of the three unknown numbers a, b, and c, that will answer the conditions

conditions of Colonel Titus's Problem, to wit, the three numbers 2.525,5 &c, 2.969,15 &c, and 3.240,5 &c; and the greatest root will enable us to find the second set of values of the said three unknown numbers, that will answer the same conditions, to wit, the three numbers 0.418,919,47, 3.912,226,8 &c, and 4.044,884,6 &c. This Solution of this Problem I have therefore inserted in this Collection, immediately after the foregoing Solution of it by Dr. Wallis, and have exhibited and explained it very fully. It begins in page 240, and ends in page 275.

In this Solution Mr. Frend puts $b = xa$, and $c = ya$; and in the course of his Solution shews that y will be $= \frac{lx^2 - m}{mx - l}$, or $\frac{16x^2 - 17}{17x - 16}$. So that, when x is discovered, the proportion of c, or ya, to a will be known, as well as the proportion of b, or xa, to a. His object in the Solution is therefore to find x. This Solution of Mr. Frend is contained in art. 23, 24, 25, and 26, and pages 240, 241, 242, 243, and 244, and produces the biquadratick equation $x^4 - \overline{\frac{ln + m^2}{mn - l^2}} \times x^3 + \frac{4lm}{mn - l^2} \times xx - \overline{\frac{l^2 + mn}{mn - l^2}} \times x = \frac{m^2 - ln}{mn - l^2}$. And in art. 27, pages 244 and 245, it is shewn that, if we substitute in this equation, instead of the letters l, m, and n, the numbers that are denoted by them, to wit, 16, 17, and 18, the said equation will be converted into the numeral equation $x^4 - 11.54 \times x^3 + 21.76 \times x^2 - 11.24 \times x =$

$x = 0.02$. This equation must therefore be resolved, in order to obtain the value of x; and from the value of x so obtained, or, if the equation should have more than one root, (as is the case,) from that value of x which has a relation to the present Problem, we must then derive the value of y by computing the fraction $\frac{lx^2 - m}{mx - l}$ or $\frac{16x^2 - 17}{17x - 16}$, to which y is equal. And, when this is done, we must multiply the value of xa, or b, into the value of ya, or c; which will give us the value of bc expressed by it's relation to aa: and then, by substituting this value of bc instead of bc in the equation $aa + bc = 16$, we shall convert the said equation into a quadratick equation involving only one unknown quantity, namely, a, and which may therefore be easily resolved. And the resolution of this quadratick equation will give us the value of a, or the first of the three unknown quantities a, b, and c, that the Problem requires us to find. And from the value of a, so found, we may derive the values of b and c by multiplying a into x to obtain the number b, and by multiplying it into y to obtain the number c. Our chief business therefore is to find the roots of the biquadratick equation $x^4 - 11.54 \times x^3 + 21.76 \times x^2 - 11.24 \times x = 0.02$, which is done very fully in pages 246, 247, 248, &c, - - 273, by the application of Mr. Raphson's method of approximation; and the first, or least, root of this equation is found (in pages 246, 247, 248, &c, - - 255,) to be very nearly $= 1.027,179,787$, which, it is shewn in art. 34, page 235, will not enable us to solve Colonel Titus's

Titus's Problem; and the fecond, or middle, root is found (in art. 35, 36, 37, &c, - - - 40, pages 256, 257, &c, - - - 265) to be nearly = 1.175,056,46; and this root is (in art. 41 and 42, pages 266 and 267,) applied to the Solution of Colonel Titus's Problem, and is found to anfwer the purpofe, by enabling us to find 2.525,534 for the value of a, or the firft of the three unknown numbers a, b, and c, which we were required to difcover, and 2.969,159 for the value of (xa, or) b, or the fecond of the faid three numbers, and 3.240,537 for the value of (ya, or) c, or the third, or laft, of the faid numbers; which three values of a, b, and c are nearly the fame with the values of them found before in the firft part of Dr. Wallis's Solution of this Problem; and, laftly, the third, or greateft, root is found (in art. 44, 45, 46, pages 268, 269, &c, - - - 273) to be nearly = 9.338,851,9; and this root is (in art. 47 and 48, pages 273, 274, and 275,) applied to the Solution of Colonel Titus's Problem, and is found to anfwer the purpofe, by enabling us to find 0.418,919,47 for the value of a, and 3.912,226,888 for the value of (xa, or) b, and 4.044,884,650 for the value of (ya, or) c; which three values of a, b, and c are nearly the fame with the latter three values of the fame quantities found before in the fecond part of Dr. Wallis's Solution of this Problem.

This Problem is therefore now compleatly folved by Mr. Frend's method of proceeding, as it had been before by that of Dr. Wallis. And Mr. Frend's Solution has this advantage over that of Dr. Wallis, that it faves us

the trouble of those very tedious and perplexing Algebräical multiplications, subtractions, and divisions, which were necessary in Dr. Wallis's Solution, and which it is very difficult to perform without making some slip, or error, either in setting down the signs + and —, that are to be prefixed to the several terms, or in computing the co-efficients of the terms, or the indexes of the powers of the quantities involved in them.

The investigations of the three roots of the biquadratick equation $x^4 - 11.54 \times x^3 + 21.76x^2 - 11.24 \times x = 0.02$ are eminent instances of the utility of Mr. Raphson's method of resolving equations by approximation, and in that view, as well as on account of their relation to Colonel Titus's Problem, were fit to receive a place in the present Volume, which is intended to illustrate and recommend that method of resolving equations.

The next Tract in the present Volume is re-printed from my Appendix to Mr. Frend's *Principles of Algebra;* and it had been published for the first time in the large octavo volume of Tracts containing a part of Mr. James Bernoulli's excellent Treatise *De Arte Conjectandi*, and other mathematical pieces, in the year 1795. It is intitled, *Observations on Mr. Raphson's Method of resolving Affected Equations of all Degrees by Approximation.* It begins in page 279, and ends in page 323, and it's contents may be described as follows. The first part of it, as far as page 303, is intended, partly, to remove some difficulties that occur in reading Mr. Raphson's excellent Treatise

Treatise on the Resolution of all Equations, (whether pure or affected,) by Approximation, intitled *Analysis Æquationum Universalis*; which difficulties are not inherent in the subject itself, or necessarily belonging to his method of resolving equations, but have arisen merely from his having unfortunately adopted the doctrine and language of *negative roots* of equations, by which the Science of Algebra, or Universal Arithmetick, has been disgraced and rendered obscure and difficult, and disgusting to men of a just taste for accurate reasoning, ever since it's introduction by Harriot and Des Cartes. The first part of this Tract is, I say, intended, partly, to remove some difficulties of this kind in the said Treatise of Mr. Raphson, and, partly, to illustrate his method of resolving high equations in other cases, or where no negative roots are mentioned, by performing the resolution of one of the equations adduced by him as examples of his method, to wit, the resolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, in a very full and distinct manner, with every step of the resolution, and the reasonings upon which it is grounded, set forth at length, agreeably to the principles laid down by him in the beginning of the said Treatise, instead of resorting (as he has done in his resolution of the same equation, and in those of all the other equations adduced by him as examples of his method,) to the repeated application of a general theorem, or canon, which he has deduced from the said principles: because I was of opinion that the latter way of performing the said resolution, by means of a theorem, or canon, affords much less satisfaction to the mind of the reader, or calculator,

in the uſe of it than he would receive by reſolving the equation in the former way, or by the immediate applica-ation of the principles themſelves, as is done in the reſolution here given of the ſaid equation. And the following part of this Tract (from page 303 to page 317,) contains ſome obſervations on the reſemblance between Mr. Raphſon's method of reſolving an equation by approximation and Sir Iſaac Newton's method of doing the ſame thing; which latter method in it's firſt proceſs, (or in finding the ſecond near value of x, or the root of the equation, after having found a, or it's firſt near value, by conjecture, or otherwiſe, to a moderate degree of exactneſs,) is preciſely the ſame with Mr. Raphſon's, but in the following proceſſes differs a little from it. This circumſtance makes it neceſſary to compare the two methods with each other, by applying them to the reſolution of the ſame equation; as we before compared Mr. Raphſon's method with Dr. Halley's, by applying them both ſucceſſively to the reſolution of the ſame three equations in the foregoing Appendix to Dr. Halley's Diſcourſe. And with this view I have (in pages 305, 306, 307, - - - 317) reſolved the cubick equation $x^3 - 2x = 5$ (which is the equation adduced by Sir Iſaac Newton as an example of his method,) by both Mr. Raphſon's method and Sir Iſaac Newton's method, in a very full and diſtinct manner, and ſo as to ſet the reſpective advantages of the two methods in oppoſition to each other, to the end that the reader may be enabled to compare them one with the other, and determine which of the two is, upon the whole, the more convenient. And here I muſt confeſs that it appears to me,

upon

upon making this comparison, that Mr. Raphson's method of approximation is, in some degree, preferable even to Sir Isaac Newton's method, though not so much as to that of Dr. Halley.

In art. 22, pages 317, 318, and 319, a proof is given of the exactness of the number 2.094,551,48, which has been found by both Mr. Raphson's and Sir Isaac Newton's methods of approximation to be the root of the equation $x^3 - 2x = 5$. And, in art. 23, pages 321, 322, and 323, there are some observations on the difficulty of finding a, or the first near value of an affected equation, in certain cases, to wit, in those cases in which the proposed equation either has, or (from the changes of the signs of it's terms from + to − and from − to +,) seems to have, more than one real and affirmative root. And with these observations this Tract concludes.

The next Tract in this Collection is the 10th Chapter of the 2nd Book of Mr. John Kersey's *Elements of Algebra*,—a most valuable work in two small volumes, folio, that was published in the year 1673. This book is now too little known, and deserves to be strongly recommended to all students of Algebra on account of the great fulness and perspicuity with which it is written; in which important qualities it very far surpasses most of the Treatises of Algebra that have been written since, excepting Dr. Saunderson's *Elements of Algebra*, in two volumes, quarto, which was published in the year 1740. And in some respects it is even preferable to Saunderson's Algebra; because it gives a fuller account of the Diophantine problems, and because it deals less in the

abſurd myſteries of *negative roots*, and *negative quantities* in general, and the doctrine of the generation of higher affected equations from lower equations by multiplication, which has been the ſource of great obſcurity and perplexity to all the ſtudents of Algebra ever ſince it's introduction into that ſcience by the publication of Harriot's Algebra in the year 1631 and Des Cartes's Geometry in the year 1637. Theſe abſurd doctrines ſeem to have delighted Dr. Saunderſon, who frequently inſults and rails at ſuch of his readers as ſhall find a difficulty in underſtanding them, as being perſons, of narrow minds and incorrigible dullneſs and ſtupidity: but Mr. Kerſey, though he did not venture boldly to reject them, as either poſitively abſurd and unintelligible or, at leaſt, leading to obſcurity, yet ſays but little about them, and ſeems deſirous, as much as poſſible, to keep clear of them. His book therefore deſerves to be ſtudied by all perſons who deſire to become ſkilfull in Algebra; and to confirm this recommendation of it I will add the following paſſage from the Preface of the learned Dr. Wallis to the Latin Edition of his Algebra, in the ſecond volume of the Collection of all his Works publiſhed at Oxford in three volumes, folio, in the year 1693, where, after mentioning other eminent writers on this ſubject, he has theſe words: *Suaſerim ut lector conſulat ex noſtris (præter alios,) Kerſæum noſtrum; qui duobus voluminibus integram Algebræ tractationem exhibuit, fusè quidem et perſpicuè traditam: quo nemo felicius quæſtiones Diophantæas elucidavit.* The chapter of this valuable book which is here reprinted, (and which I had reprinted already about a year ago in the Appendix to Mr. Frend's *Principles of Algebra*,) relates to the method invented by Simon

Stevinus, the Flemiſh Mathematician, (who flouriſhed in the beginning of the laſt century, and died in the year 1633,) for finding a, or the firſt near value of x, or the root of any propoſed Algebräick equation, by repeated conjectures and trials with eaſy numbers of one or two decimal figures: after having found which we may proceed to determine the value of the ſaid root to a greater degree of exactneſs by one or more applications of Mr. Raphſon's method of approximation. The title of this 10th Chapter of Mr. Kerſey's Algebra is as follows: *An Explication of Simon Stevin's General Rule to extract one Root out of any poſſible Equation in Numbers, either exactly or very nearly true.*

I have next inſerted in this Collection of Tracts a Remark, (which, as well as the two preceeding Tracts, I had printed before in my Appendix to Mr. Frend's *Principles of Algebra,*) on a groſs error in the reaſoning of the late celebrated French Mathematician, Monſieur Clairaut, in his elegant little treatiſe, intitled *Elémens d'Algébre*, where he endeavours to prove that, if two negative quantities, called $-b$ and $-d$, be multiplied together, their product will be $+bd$. This Propoſition he ſhould have, firſt, made intelligible, by telling us what he meant by a negative quantity, before he undertook to demonſtrate it. But this he has not done. For, in the beginning of his book, he ſpeaks of the ſign $-$, or *minus*, as being only the ſign of the ſubtraction of the quantity to which it is prefixed from another quantity which is greater than it; according to which meaning of the ſign $-$, (which is very clear and juſt) the compound quantity $a-b$ muſt ſignify the exceſs of

the quantity a above the quantity b, which is supposed to be less than a. He then proves clearly and by just reasonings that, if the compound quantity $a - b$ is multiplied into the compound quantity $c - d$, the product of the said multiplication will be the quadrinomial quantity $ac - bc - ad + bd$. But then he supposes a and c to become equal to 0, while b and d retain their former magnitudes, not considering that, according to the only idea which he had before given us of the sign $-$, the quantity a must be greater than b, and the quantity c must be greater than d, in order to make it possible for the compound quantities $a - b$ and $c - d$ to exist; and then he concludes that, since a and c are become equal to 0, the three terms ac, $-bc$, and $-ad$, (which involve them) will become equal to 0 likewise, and the whole quadrinomial quantity $ac - bc - ad + bd$ will become equal to it's last term $+ bd$, and consequently that $\overline{0 - b} \times \overline{0 - d}$, or $- b \times - d$, will be equal to $+ bd$. This is very weak reasoning for so great a Mathematician as Monsieur Clairaut: and, if he, who was so justly celebrated for the clearness of his style and the accuracy of his reasonings on many occasions, could offer nothing better than this fallacious argument in support of this favourite doctrine of negative quantities, or quantities less than nothing, or quantities arising from the subtraction of some quantities from others that are less than themselves (which are the definitions given of these quantities by Sir Isaac Newton, Dr. Saunderson, and Mr. MacLaurin, as is shewn in the Note in page 286 of the present Volume,) I think we may fairly conclude that this doctrine cannot be defended:

Si

Si Pergama dextrâ
Defendi possent, etiàm hâc defensa fuissent.

This Remark begins in page 339, and ends in page 343.

The next Tract in this Collection is intitled, *A General Method of investigating the two, or three, first Figures of the least Root of an Equation that has more than one real and affirmative Root;* and is reprinted from the 3d Volume of the *Scriptores Logarithmici*, pages 725, 726, &c, - - - - 761.

The foregoing method of finding a, or the first near value of the root of a proposed equation, by means of conjectures and trials with easy numbers of only one or two figures, in the manner recommended by Stevinus and Kersey, (which is set forth in the 10th chapter re-printed from Kersey's Algebra in pages 325, 326, &c, - - - 336 of this Volume,) will be found perfectly convenient in all equations that have but one root. But, when the proposed equation has, or (from the changes of the signs + and −, prefixed to it's terms, from + to — and from — to +,) seems to have, two, or three, or four, or more, roots, or (in the language of modern Algebräists,) real and affirmative roots, we may sometimes be at a loss to know which of the said several roots we ought to go first in pursuit of. In these cases therefore I am inclined to think that it will be most prudent, in general, to begin with the investigation of the least root. And the manner of doing this is the subject of this Tract. The proposition on which this method of proceeding

proceeding is founded, is as follows: to wit, "that, if all the terms of the propofed equation that have the fign — prefixed to them are expunged from it, the root of the equation fo curtailed will be lefs than the leaft root of the original equation;" whence it follows that, if this root is found to two places of figures by Stevinus's method of conjecture and trial, or otherwife, and be called a, it will ferve for the ground-work of an approximation to the value of the leaft root of the original equation in Mr. Raphfon's method by putting the faid leaft root equal to $a + z$, and fubftituting $a + z$ inftead of x in the terms of the original equation, and refolving the tranfformed equation refulting from this fubftitution, as if it were a mere fimple equation, in the manner recommended by Mr. Raphfon. The principle, or propofition above-mentioned, upon which this method is grounded, is defcribed and proved in the Lemma fet forth and demonftrated in pages 361, 362, 363, &c, - - - 367; and fome examples are given of it in pages 368, 369, &c, - - - 431, which are refolved at great length and with great care and labour: and methods are pointed-out by which, when we have obtained the leaft, or, perhaps, the only, root of the propofed equation, we may, in the firft place, difcover whether the faid equation has, or has not, any other root befides that which has been already found, and, fecondly, if the faid equation has any other root, we may reduce the equation to another equation of the next lower order, or degree, of which the roots will be the fame with the remaining roots of the original equation.

After having dwelt fo long upon Dr. Halley's, Mr. Raphfon's, and Sir Ifaac Newton's, methods of refolving equations

equations by approximation, I conceived that it would be agreeable to the readers of these Tracts to be made acquainted with Vieta's method of resolving them also by approximation, which was invented by that celebrated author above a hundred years before the publication of Mr. Raphson's and Dr. Halley's methods, and which was generally adopted and practised by the ablest Mathematicians of those times, and particularly by Mr. Oughtred, Dr. Pell, and Dr. Wallis, though it has since been abandoned on account of the greater facility and expedition of the methods of Sir Isaac Newton, Mr. Raphson, and Dr. Halley. I have therefore added another Tract to those already mentioned, in which I have endeavoured to explain the principles upon which Vieta's method of resolving equations by approximation is grounded, and the manner in which we are enabled by it to investigate the value of the root of any equation whatsoever. I have, however, confined myself, in this attempt, to the resolution of only one affected equation by this method, as a specimen of it, to wit, the equation $x^5 - 5x^3 + 500x = 7,905,504$, which is the highest and most complicated of all the affected equations which Vieta has adduced as examples of his method of resolution, and which is resolved in the 15th Problem of his Discourse on this subject, which he intitles *De Numerosâ Potestatum Adfectarum Resolutione.* But, as Vietá's method of resolving *affected* equations by approximation is similar to, and derived from, his method of resolving *pure* equations of the same degree, or of extracting the roots of numbers, I have begun my explanation of his method of resolving the affected equation $x^5 - 5x^3 + 500x = 7,905,504$ by, first, shewing how

how he would resolve the pure equation $x^5 = 7,905,504$, or extract the fifth root of the number 7,905,504. This is done very fully and distinctly in art. 4, 5, 6, and 7, pages 441, 442, &c, - - - - 451; and the result is, that the above fifth root is = 23.96 &c. I then (in art. 9, pages 452, 453) compare this result with the value of the same fifth root obtained by means of Monsieur de Lagny's general expression for the mth root of any given number N, upon a supposition that a, or a first near value of it, is already known. This expression has been given in the Note in page 9 of this Volume, and is

$$a + 2a \times \frac{N - a^m}{\overline{m-1} \times N + \overline{m+1} \times a^m}, \text{ or, when } m \text{ is } = 5,$$

(as it is in the present case,) $a + 2a \times \frac{N - a^5}{4N + 6a^5}$, or

$a + a \times \frac{N - a^5}{2N + 3a^5}$; which, if a is = 23.9, will be (= 23.9 + 0.065,466,9, or) 23.965,466,9; of which number the first eight figures, 23.965,466, are, probably, exact: So that no less than five new figures, to wit, the figures 0.065,466, are obtained exactly by means of this expression of Monsieur de Lagny, from the first three figures, 23.9, that were already known; whereas by Vieta's method of proceeding only one new figure, to wit, the figure 0.06, was obtained by the corresponding process from the same three figures, 23.9, that were already known. I then compare the same result of Vieta's method of extraction, to wit, the number 23.96, with the value of the same fifth root of the same number 7,905,504 obtained by means of Mr. Raphson's simpler expression for the mth root of any given number N, upon a supposition that a, or a first near value of it, is already known,

known, to wit, the expression $a + \frac{N - a^m}{ma^{m-1}}$, or, when m is $= 5$, (as it is in the present case,) $a + \frac{N - a^5}{5a^4}$; and I find that, if a is supposed to be $= 23.9$, this expression will be $=$ ($23.9 + 0.065,8$, or) $23.965,8$, which agrees with the more accurate value of the same fifth root found before by Monsieur de Lagny's expression, to wit, 23.965,466,9, in the five first figures 23.965, which therefore must be exact: so that even this simple expression of Mr. Raphson has given us two new figures of the fifth root sought, to wit, the two figures 0.065, exactly, in addition to the first three figures 23.9, which were already known; whereas by Vieta's method of proceeding only one new figure, to wit, the figure 0.06, was obtained by the corresponding process in addition to the same three figures 23.9, that were already known.

By these comparisons it appears that both Mr. Raphson's and Monsieur de Lagny's expressions for the value of the fifth root of the given number 7,905,504 have greatly the advantage of Vieta's method of extracting the same root in point of expedition, though not in point of accuracy of reasoning and certainty of the conclusions obtained by them; in which respects they are all equally valuable.

In art. 11, pages 454, 455, and 456, I have pointed-out the resemblance between Vieta's method of extracting the fifth root of the number 7,905,504, and Mr. Raphson's method of extracting it, and shewn that they

are

are almost the same. For they both proceed by a division of the same number, or dividend, 107,391.348,01, by the same divisor 1,631,404.3205; which occurs first in page 449, in extracting the fifth root of 7,905,504 by Vieta's method, and afterwards in page 454 in finding the same fifth root by computing Mr. Raphson's expression $a + \frac{N - a^m}{ma^{m-1}}$: and the only difference between them is that Mr. Raphson continues the division to three figures in the quotient, to wit, the three figures 0.0658, and thereby obtains 23.9 + 0.0658, or 23.9658, for the next near value of the fifth root sought, of which number all the figures, except the last figure 8, are exact; whereas Vieta carries the division to only one figure in the quotient, to wit, the figure 0.06, and thereby obtains only 23.9 + 0.06, or 23.96, for the next near value of the fifth root sought. If Vieta had happened to observe that the two first figures of the quotient 0.0658 would have been exact, and that, in general, if n is the number of figures that are exact in a, or the part of the root that is already known, the first $n - 1$ figures of the quotient would be exact, and had accordingly carried his division to that number of figures in the quotient, his method of extracting the fifth, or the mth, root of any given number N would have been exactly the same with Mr. Raphson's.

Having thus gone through the extraction of the fifth root of the number 7,905,504, or the resolution of the pure equation $x^5 = 7,905,504$, in the manner in which Vieta had directed such resolutions of pure equations to be

be performed; I proceed (in art. 12, 13, 14, and 15, pages 456, 457, 458, &c, - - - 463,) to exhibit Vieta's method of resolving the affected equation $x^5 - 5x^3 + 500x = 7,905,504$, which is similar to the foregoing method of resolving the pure equation $x^5 = 7,905,504$. And by this resolution it appears that the root x of that equation is exactly $= 24$. I then (in art. 16, page 463,) resolve the same equation $x^5 - 5x^3 + 500x = 7,905,504$ by Mr. Raphson's method of approximation, in order to compare the processes of this resolution with those in Vieta's resolution just before exhibited. And I find that there is the same resemblance between these two methods of resolving this affected equation that I had before observed to take place between the same two methods of resolving the pure equation $x^5 = 7,905,504$. For here, as before, both methods of resolution lead us to the division of the same dividend, to wit, the number 4,735,504, by the same divisor, 794,500. But Vieta carries the division to only one figure in the quotient, which in the present example is sufficient for his purpose, because this equation has the whole number 24, consisting of only two figures, for it's exact root, but which, if the root were an irrational quantity, or incommensurable to unity, (as it most commonly is,) and were required to be found to ten, or twelve, places of figures, would make his method of resolution excessively tedious and laborious; whereas Mr. Raphson continues his divisions to as many figures in the quotients, or to as many wanting one, as there are figures that are exact in a, or the part of the root x that is already known; by which means, if the figures that are exact in a, or the known part

part of the root, are five, or six, or seven, he will obtain four, or five, or six, new figures in the quotient to be added to a, that will be all exact. And in this circumstance alone consists the advantage of Mr. Raphson's method of approximation over that of Vieta. If Vieta had happened to observe that these divisions might be safely continued to several figures in the quotients, or that several figures in the quotients, (namely, as many, wanting one, as there are figures that are exact in a, or the part of the value of x that is already known,) would be exact, and had consequently directed his readers to continue the divisions to that number of figures in the quotients, his method would, as I conceive, have been the very same with Mr. Raphson's, and the art of resolving all sorts of equations by approximation would not only have been invented by him (as it has been,) but would have been brought by him at once to the highest degree of perfection of which it, probably, is capable. He therefore ought to be considered as the original founder of the whole doctrine of resolving equations by approximation, though Sir Isaac Newton, Mr. Raphson, Monsieur de Lagny, and Dr. Halley, and, perhaps, some subsequent Mathematicians, have made valuable improvements on his method, by which the practice of it has been very much facilitated.

This Specimen of Vieta's Method of resolving Equations by approximation begins in page 435, and ends in page 470.

The last Tract in this Collection is written by the learned and ingenious Mr. Frend, the author of a valuable

able Treatiſe of Algebra in one volume, octavo, intitled, *Principles of Algebra*, in which he has totally diſcarded the obſcure and myſterious doctrines of negative quantities, and impoſſible, or imaginary, roots of equations, and the generation of higher equations from lower ones by bringing all the terms of the lower equations to the ſame ſide of the equation, ſo as to make the reſults of them equal to nothing, and then multiplying them, (when thus made equal to o,) into each other, and thereby producing equations of higher orders that will likewiſe be equal to nothing. His preſent Tract, which is intitled "*Remarks on the Number of Negative and Impoſſible Roots in Algebräick Equations*," has the ſame laudable tendency to free Algebra from obſcurity, by ſhewing more clearly than is uſually done by writers on this ſubject, what is meant by *the negative roots* of an equation, to wit, "that they are in truth the roots, or (in the language of modern Algebräiſts,) the *affirmative*, or *poſitive*, roots, of another equation which is derived from the firſt equation by changing the ſigns + and — that are prefixed to ſuch of it's terms as involve any odd powers of the unknown quantity x, into the contrary ſigns," and by obſerving that *impoſſible roots* are mere fictitious quantities, of which no clear idea can be formed, which have been introduced into Algebra to make-up the number of roots which *Harriot* and *Des Cartes* and their followers have thought proper to aſcribe to every equation of any given degree, or order, denoted by any whole number called m, to wit, m roots; though, in truth, there is only one form in every degree, or order, of equations, (as, for example, the form $px - x^2 = q$ in quadratick equations,

and the form $x^3 - px^2 + qx = r$ in cubick equations, and the form $rx - qx^2 + px^3 - x^4 = s$ in biquadratick equations,) that can really have fo many roots. When therefore an equation occurs, of the mth order, that has fewer than m roots, the modern Algebräifts firft call-in *the negative* roots of the equation, (or *the real and affirmative* roots of the fecond equation which is derived from the firft equation in the manner above-defcribed,) in aid of the former, or real and affirmative, roots of it; and, if the fum of both the affirmative and the negative roots of the equation, (which fum we will fuppofe to be denoted by the letter n,) is likewife lefs than m, (which is often the cafe,) they then declare that, befides the n real roots, both affirmative and negative, the equation has $m - n$ *impoffible*, or *imaginary*, roots, which, being added to the former n roots, makes up the full number of m roots, which they had before declared that the faid equation ought to have. Thefe Remarks begin in page 473, and end in page 479. And with thefe Remarks the prefent Volume concludes.

A Note

A Note on Page 242.

IN page 242, &c, Mr. Frend's Solution of Colonel Titus's Problem may be made ſhorter and eaſier by making the following changes in it.

In art. 25, lines 6 and 7, we have $myy - ny = nxx - mx$. From this equation let the Solution be as follows.

But it has been ſhewn in art. 24 that y is $= \frac{lx^2 - m}{mx - l}$.

Therefore yy will be $\left(= \frac{\overline{lx^2 - m}^2}{\overline{mx - l}|^2}\right) = \frac{l^2x^4 - 2lmx^2 + m^2}{\overline{mx - l}|^2}$, and myy will be $= \frac{l^2mx^4 - 2lm^2x^2 + m^3}{\overline{mx - l}|^2}$; and ny will be $= \frac{lnx^2 - mn}{mx - l} = \frac{\overline{lnx^2 - mn} \times \overline{mx - l}}{\overline{mx - l}|^2} = \frac{lmnx^3 - l^2nx^2 - m^2nx + lmn}{\overline{mx - l}|^2}$; and conſequently $myy - ny$ will be $= \frac{l^2mx^4 - 2lm^2x^2 + m^3}{\overline{mx - l}|^2}$ − the fraction $\frac{lmnx^3 - l^2nx^2 - m^2nx + lmn}{\overline{mx - l}|^2} = \frac{l^2mx^4 - lmnx^3 - 2lm^2x^2 + l^2nx^2 + m^2nx + m^3 - lmn}{\overline{mx - l}|^2}$.

But $myy - ny$ is $= nxx - mx$.

Therefore

Therefore the fraction juſt now obtained, to wit,

$$\frac{l^2mx^4 - lmnx^3 - 2lm^2x^2 + l^2nx^2 + m^2nx + m^3 - lmn}{\overline{mx-l}|^2}$$

will alſo be $= nxx - mx$; and (multiplying both ſides into $\overline{mx-l}|^2$,) the ſeptinomial quantity $l^2mx^4 - lmnx^3 - 2lm^2x^2 + l^2nx^2 + m^2nx + m^3 - lmn$ will be ($= \overline{nxx - mx} \times \overline{mx-l}|^2 = \overline{nxx - mx} \times \overline{m^2x^2 - 2lmx + l^2}$) = the ſextinomial quantity $m^2nx^4 - 2lmnx^3 + l^2nx^2 - m^3x^3 + 2lm^2x^2 - l^2mx$. Therefore (adding $lmnx^3 + 2lm^2x^2$ to both ſides,) we ſhall have $l^2mx^4 + l^2nx^2 + m^2nx + m^3 - lmn = m^2nx^4 - lmnx^3 + l^2nx^2 - m^3x^3 + 4lm^2x^2 - l^2mx$, and (ſubtracting $l^2mx^4 + l^2nx^2 + m^2nx$ from both ſides,) $m^3 - lmn = m^2nx^4 - l^2mx^4 - lmnx^3 - m^3x^3 + 4lm^2x^2 - l^2mx - m^2nx$, and (dividing all the terms by m,) $m^2 - ln = mnx^4 - l^2x^4 - lnx^3 - m^2x^3 + 4lmx^2 - l^2x - mnx = \overline{mn - l^2} \times x^4 - \overline{ln + m^2} \times x^3 + 4lm \times x^2 - \overline{l^2 + mn} \times x$, and (dividing all the terms by $mn - l^2$, the co-efficient of x^4, the higheſt term,) $x^4 - \frac{ln + m^2}{mn - l^2} \times x^3 + \frac{4lm}{mn - l^2} \times x^2 - \frac{l^2 + mn}{mn - l^2} \times x = \frac{m^2 - ln}{mn - l^2}$; which is the final equation obtained, with more difficulty, at the end of art. 26, page 244.

This improvement of Mr. Frend's Solution was ſuggeſted to me by my learned friend, the Reverend Mr. John Hellins, B. D. Vicar of Potter's Pury in Northamptonſhire.

E R-

ERRATA.

IN page 7, line 4, instead of 6.358, read 6.1358.

In page 8, line 11, after the word *had*, insert *been*.

In page 10, line 11, under the radical sign, instead of $\frac{1}{35}aa$, read $\frac{1}{36}aa$.

In page 12, line 18, after the word *column*, dele *to*.

In page 16, line 6, instead of 10.26, read − 10.26.

In page 66, line 13, instead of *consequently*, read *considerably*.

In page 114, line 2 from the bottom, instead of zaz, read $2az$.

In page 139, line 12, instead of $17d$, read $17d^2$.

And in the same page 139, line 16, instead of $17d$, read $17d^2$.

In page 161, lines 7 and 9, instead of 0.000,000,045,866,14, and 0.350,987,045,866,14, read 0.000,000,045,866,06, and 0.350,987,045,866,06. For, upon a re-examination of the computation in page 431, which makes $z = 0.000,000,045,866,06$, and not $= 0.000,000,045,866,14$, I have found it to be right.

In page 168, line 7 from the bottom, before $\overline{34.8}|^2$, insert the mark $=$.

In page 204, line 11 from the bottom, instead of $168m4na^8$, read $168m^4na^8$.

In page 212, line 5, instead of $4l^4mna^2$, read $4lm^2n^2a^2$.

In page 218, line 9, instead of $+ mne^8$, read $- mne^8$.

In page 227, line 14, instead of $7le^2$, read $7l^3e^2$.

And in the same page 227, line 4 from the bottom, instead of $7le^2$, read $7l^3e^2$.

In

In page 247, line 6, inſtead of 11.24, read 11.24*x*.

And in the ſame page 247, line 2 from the bottom, inſtead of 0.2, read 0.02.

In page 249, line 3, inſtead of 1.112,453,441, read 1.112,453,263,441.

In page 297, line 3 from the bottom, inſtead of *xx*, read *x*.

In page 353, line 19, inſtead of *the*, read *thoſe*.

And in the ſame page 353, lines 2 and 4 from the bottom, after the number 14,937, and after the word *increaſes*, inſert a comma.

In page 390, line 3, before the number 12,721.120, inſert the mark =.

In page 393, line 15, under the vinculum, inſtead of $ba + ba$, read $b^2a + ba^2$.

In page 404, line 2 from the bottom, after the word *equal*, inſert *to*.

In page 430, line 4 from the bottom, inſtead of −, read +.

In page 449, line 7, inſtead of $5 \times 23 \times 0{\cdot}9$, read $5 \times 23 \times 0{\cdot}6561$.

In page 454, line 12, inſtead of $\sqrt{5}\overline{7,905,504}$ read $\sqrt[5]{7,905,504}$.

In page 455, line 15, inſtead of 1,631,404,3205 read 1,631,404.3205.

A Table

A Table of the Contents of this Collection of Mathematical Tracts.

NUM-

NUM.

NUMBER VI.

NUMBER VII.

NUMBER VIII.

NUM-

NUMBER IX.

NUMBER X.

A NEW,

A NEW, EXACT AND EASIE METHOD

OF

FINDING THE ROOTS OF ANY EQUATIONS GENERALLY, AND THAT WITHOUT ANY PREVIOUS REDUCTION.

By DR. EDMUND HALLEY.

A New, Exact and Easie METHOD *of finding the* ROOTS *of any* EQUATIONS *Generally, and that without any previous Reduction. By* EDM. HALLEY. *Being Number* 210 *of the* Philosophical Transactions *published in May* 1694.

THE principal use of the *Analytick Art*, is to bring Mathematical Problems to Equations, and to exhibit those Equations in the most simple Terms that can be. But this Art would justly seem in some degree defective, and not *sufficiently Analytical*, if there were not some Methods, by the help of which the Roots (be they Lines or Numbers,) might be gotten from the Equations that are found, and so the Problems in that respect be solved. The Ancients scarce knew any thing in these Matters, beyond *Quadratick Equations*. And what they writ of the *Geometrick Construction* of solid Problems, by the help of the *Parabola*, *Cissoid*, or any other Curve, were only particular things design'd for some particular Cases. But as to *Numerical Extraction*, there is every where a profound Silence; so that whatever we perform now in this kind, is entirely owing to the Inventions of the Moderns.

And first of all, that great Discoverer and Restorer of the Modern Algebra, *Francis Vieta*, about 100 years

since,

ſince, ſhew'd a general Method for extracting the Roots of any Equation, which he publiſh'd under the Title of, *A Numerical Reſolution of Powers*, &c. And the ſubſequent writers *Harriot*, *Oughtred*, and others, as well of our own Country as Foreigners, ought to acknowledge whatſoever they have written upon this Subject, as taken from *Vieta*. But what the Sagacity of Mr. *Newton*'s* Genius has perform'd in this buſineſs, we may rather conjecture (than be fully aſſured of) from that ſhort Specimen given by Dr. *Wallis* in the 94th Chapter of his *Algebra*. And we muſt be forced to expect it, till his great Modeſty ſhall yield to the Intreaties of his Friends, and ſuffer thoſe curious Diſcoveries to ſee the Light.

Not long ſince (*viz. A. D.* 1690) that excellent Perſon M. *Joſeph Raphſon*, F.R.S. publiſh'd his *Univerſal Analyſis of Equations*, and illuſtrated his Method by plenty of Examples; by all which he has given Indications of a Mathematical Genius, from which the greateſt things may be expected.

By his Example, M. *de Lagny*, an ingenious Profeſſor of Mathematicks at *Paris*, was encouraged to attempt the ſame Argument; but he, being almoſt altogether taken up in extracting the Roots of pure Powers (eſpecially the Cubick) adds but little about Affected Equations, and that pretty much perplex'd too, and not ſufficiently demonſtrated. Yet he gives two very compendious Expreſſions for the Approximation of a Cubical Root; one a Rational, and the other an Irrational one. Ex. gr. that the ſide of the cube $aaa + b$ is between

* Afterwards Sir Iſaac Newton.

$$a + \frac{ab}{3aaa + b}, \text{ and } \sqrt{\tfrac{1}{4}aa + \frac{b}{3a}} + \tfrac{1}{2}a.$$

And the root of the 5th Power $a^5 + b$, he makes $= \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$ (where Note, that 'tis $\frac{1}{4}aa$, not $\frac{1}{2}aa$, as 'tis erroneoufly printed in the French Book.) Thefe Rules were communicated to me by a Friend, I having not feen the Book; but, having by tryal found the goodnefs of them, and admiring the Compendium, I was willing to find out the Demonftration: Which having done, I prefently found that the fame Method might be accommodated to the Refolution of all forts of Equations. And I was the rather inclin'd to improve thefe Rules, becaufe I faw that the whole thing might be explain'd in a *Synopfis*; and that by this means, at every repeated ftep of the Calculus, the Figures already found in the Root would be at leaft trebled; which, by all other ways*, are encreafed but in an equal Number with the given ones. Now, the fore-mention'd Rules are eafily demonftrated from the Genefis of the Cube, and the 5th Power. For, fuppofing the fide of any Cube to be $= a + e$, the Cube arifing from thence will be $aaa + 3aae + 3aee + eee$. And confequently, if we fuppofe aaa to be the next lefs Cube to any given Non-cubick Number denoted by $aaa + b$, and $a + e$ to be the Cube-root of the faid Number, it is evident that e will be lefs than 1, and confequently that eee will alfo be lefs than 1, or Unity, and that the remainder b, or the difference of the Numbers aaa and $aaa + b$, will be $=$ the three other Members of the Cube, to wit, $3aae +$

* This alludes to Mr. Raphfon's and Sir Ifaac Newton's Methods.

$3aee + eee$. Whence, rejecting eee upon the account of its smallness, we have $b = 3aae + 3aee$. And, since aae is much greater than aee, the quantity $\frac{b}{3aa}$ will not much exceed e; so that, putting $e = \frac{b}{3aa}$, then the quantity $\frac{b}{3aa + 3ae}$ (to which e is nearly equal) will be found

$$= \frac{b}{3aa + \frac{3ab}{3aa}}, \text{ or } \frac{b}{3aa + \frac{b}{a}}; \text{ that is, } \frac{ab}{3aaa + b} \text{ will be}$$

nearly $= e$. And so the side of the Cube $aaa + b$ will be nearly $= a + \frac{b}{3aa + \frac{b}{a}}$, or $a + \frac{ab}{3aaa + b}$, which is the *Rational Formula* of *M. de Lagny*. But now, if aaa were the next greater Cubick Number to that given, the side of the Cube $aaa - b$, will, after the same manner, be found to be nearly $= a - \frac{b}{3aa - \frac{b}{a}}$, or $a - \frac{ab}{3aaa - b}$. And this easy and expeditious Approximation to the Cubick Root is (only a very small matter) erroneous in point of *defect*, the quantity e, the remainder of the Root thus found, coming something less than really 'tis *.

* I believe the expressions $a + \frac{b}{3aa + \frac{b}{a}}$, and $a - \frac{b}{3aa - \frac{b}{a}}$ will be found easier to compute to any given number of figures than the equivalent and, seemingly, more simple expressions $a + \frac{ab}{3aaa + b}$ and $a - \frac{ab}{3aaa - b}$, and more especially when a consists of four, or more, decimal figures. F. M.

As

As for the *Irrational Formula*, 'tis deriv'd from the ſame Principle, *viz.* $b = 3aae + 3aee$, or $\frac{b}{3a} = ae + ee$, and ſo $\sqrt{\frac{1}{4}aa + \frac{b}{3a}} = \frac{1}{2}a + e$, and $\sqrt{\frac{1}{4}aa + \frac{b}{3a}} + \frac{1}{2}a = a + e$, the Root ſought. Alſo the ſide of the Cube $aaa - b$, after the ſame manner, will be found to be $\frac{1}{2}a + \sqrt{\frac{1}{4}aa - \frac{b}{3a}}$. And this *Formula* comes ſomething nearer to the Scope, being erroneous in point of *exceſs*, as the other was in *defect*, and is more accommodated to the ends of Practice, ſince the Reſtitution of the Calculus, is nothing elſe but the continual addition or ſubtraction of the Quantity $\frac{eee}{3a}$ according as the quantity e can be known. So that we ſhould rather write $\sqrt{\frac{1}{4}aa + \frac{b - eee}{3a}} + \frac{1}{2}a$, in the former caſe, and, in the latter, $\frac{1}{2}a + \sqrt{\frac{1}{4}aa + \frac{eee - b}{3a}}$. But by either of the two *Formula*'s, the Figures already known in the Root to be extracted, are at leaſt tripled; which I conclude will be very grateful to all the Students in Arithmetick; and I congratulate the Inventor upon the account of his Diſcovery.

Examples of the Uſe of theſe Rules in the Extraction of the Cube-Roots of Numbers.

But that the uſe of theſe Rules may be the better perceiv'd, I think it proper to ſubjoin an Example or two.

Let it be propos'd to find the ſide of the double Cube, or let $aaa + b$ be $= 2$. Here a is $= 1$, and conſequently a^3 is $= 1$, and b is $= 1$, and $\frac{a}{2}$ is $(= \frac{1}{2}) = 0.50$, and $\frac{b}{3a}$ is $= \frac{1}{3}$, and $\frac{1}{4}aa + \frac{b}{3a}$ is $(= \frac{1}{4} \times 1 + \frac{1}{3} = \frac{1}{4} + \frac{1}{3}) = \frac{7}{12}$, and $\sqrt{\frac{1}{4}aa + \frac{b}{3a}}$ is $(= \sqrt{\frac{7}{12}} = \sqrt{0.5833}) = 0.76$, and $\frac{a}{2} + \sqrt{\frac{1}{4}aa + \frac{b}{3a}}$ is $(= 0.50 + 0.76) = 1.26$, which will be found to be the true ſide nearly. Now, the Cube of 1.26, is $2.000,376$, and ſo $0.63 + \sqrt{.3969 - \frac{.000,376}{3.78}}$, or $0.63 + \sqrt{.396800,529100,5291} = 1.259,921,049,895$ —; which in 13 Figures, gives the ſide of the double Cube, with very little trouble, *viz.* By only one diviſion, and the extraction of the ſquare Root; when-as by the common way of working, how much pains it would have coſt, the Skilful very well know. This Calculus a Man may continue as far as he pleaſes, by encreaſing the Square by the addition of the quantity $\frac{eee}{3a}$; which Correction, in this caſe, will give only the encreaſe of Unity in the 14th Figure of the Root.

Examp. 2*d.* Let it be propos'd to find the ſide of a Cube equal to that Engliſh Meaſure commonly call'd a Gallon, which contains 231 ſolid Ounces *. The next leſs Cube is 216, whoſe ſide $6 = a$, and the remainder is $15 = b$; and ſo, for the firſt Approximation, we have

* Or Cubick Inches.

$\frac{a}{2} + \sqrt{\frac{aa}{4} + \frac{b}{3a}}$ $\left(= \frac{6}{2} + \sqrt{\frac{36}{4} + \frac{15}{3 \times 6}} = 3 + \sqrt{9 + \frac{5}{6}} = 3 + \sqrt{9 + 0.8333}\right) = 3 + \sqrt{9.8333}$ = the Root. And ſince $\sqrt{9.8333\ldots}$ is $3.1358\ldots$, 'tis plain that $6.358 = a + e$. Now, let $6.1358 = a$; and we ſhall then have for its Cube - - 231.000,853,894,712, and, according to the Rule, 3.0679 $+ \sqrt{9.412{,}010{,}41 - \frac{0.000{,}853{,}894{,}712}{18.4074}}$ is moſt accurately equal to the ſide of the given Cube, which within the ſpace of an Hour, I determin'd by Calculation to be 6.135,792,439,661,958,97, which is exact in the 17th Figure, and defective in the 18th. And this *Formula* is deſervedly preferable to the *Rational Formula*, upon the account of the great Diviſor $3aaa + b$, or $3aaa - b$, in the Rational Formula, which is not to be managed without a great deal of Labour; whereas the extraction of the ſquare-root proceeds much more eaſily, as manifold Experience has taught me.

Of the Extraction of the Fifth Root of a Given Number.

But the Rule for the Root of a pure Surſolid, or the 5th Power, is of ſomething a higher Enquiry, and does much more perfectly yet, do the buſineſs: for it does at leaſt Quintuple the given Figures in the Root; neither is the Calculus very large or operoſe. Tho' the Author no-where ſhews his method of Invention, or any Demonſtration, altho' it ſeems to be very much wanting; eſpecially ſince all things are not right in the printed Book, which may eaſily deceive the Unſkilful. Now the 5th power of the ſide $a + e$ is compos'd of theſe Members, $a^5 + 5a^4e + 10a^3e^2 + 10a^2e^3 + 5ae^4 + e^5 = a^5 + b$; from

from whence b is $= 5a^4e + 10a^3e^2 + 10a^2e^3 + 5ae^4$, rejecting e^5 becaufe of its fmallnefs. Whence $\frac{b}{5a} = a^3e + 2a^2e^2 + 2ae^3 + e^4$, and (adding on both fides $\frac{1}{4}a^4$, and then extracting the fquare-roots of both fides,) we fhall have $\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} = \sqrt{\frac{1}{4}a^4 + a^3e + 2a^2e^2 + 2ae^3 + e^4} = \frac{1}{2}aa + ae + ee$. Then (fubtracting $\frac{1}{4}aa$ from both fides, and extracting the fquare-roots of the remainders,) $\frac{1}{2}a + e$ will be $= \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$; to which if $\frac{1}{2}a$ be added, then will $a + e = \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$ = the fifth root of the Number $a^5 + b$. But, if it had $a^5 - b$ (the quantity a being too great) the Rule would have been thus, $a + e = \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 - \frac{b}{5a}} - \frac{1}{4}aa}$. And this Rule approaches wonderfully, fo that there is hardly any need of Reftitution.

Of the Extraction of all other Roots of Given Numbers by Means of fimilar Rules to thofe above given for the Extraction of the Third and Fifth Roots.

But, while I confidered thefe things with my felf, I lit upon a General Method for the *Formula*'s of all Powers whatfoever, and (which being handfome and concife enough) I thought I would not conceal it from the Publick.

Thefe Formula's, (as well the *Rational*, as the *Irrational* ones) are thus.

$\sqrt{aa + b}$

$$\sqrt{aa+b} = \sqrt{aa+b}, \text{ or } a + \frac{ab}{2aa+\frac{1}{2}b}.$$

$$\sqrt[3]{a^3+b} = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + \frac{b}{3a}}, \text{ or } a + \frac{ab}{3aaa+b}.$$

$$\sqrt[4]{a^4+b} = \tfrac{2}{3}a + \sqrt{\tfrac{1}{9}aa + \frac{b}{6aa}}, \text{ or } a + \frac{ab}{4a^4+\frac{3}{2}b}.$$

$$\sqrt[5]{a^5+b} = \tfrac{3}{4}a + \sqrt{\tfrac{1}{16}aa + \frac{b}{10a^3}}, \text{ or } a + \frac{ab}{5a^5+2b}.$$

$$\sqrt[6]{a^6+b} = \tfrac{4}{5}a + \sqrt{\tfrac{1}{25}aa + \frac{b}{15a^4}}, \text{ or } a + \frac{ab}{6a^6+\frac{5}{2}b}.$$

$$\sqrt[7]{a^7+b} = \tfrac{5}{6}a + \sqrt{\tfrac{1}{36}aa + \frac{b}{21a^5}}, \text{ or } a + \frac{ab}{7a^7+3b}\text{*}.$$

And ſo alſo of the other higher Powers. But if a were aſſumed bigger, than the Root ſought (which is done

* The general expreſſions for theſe values of the root of any given number are as follows. Let the given number be called N, and (m being any whole number whatſoever,) let a near value of it's mth root that is leſs than the truth be called a. Then will the mth root of the given number N be very nearly equal either to the rational expreſſion $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, or to the irrational expreſſion $a + \sqrt{\left(\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}\right)} - \frac{a}{m-1}$; as is ſhewn at large in my Explanation of Monſieur de Lagny's method of extracting the Roots of Numbers, publiſhed with other Mathematical Tracts in a large octavo volume, in the year 1795, pages 507, 508, &c, 516. F. M.

with

with ſome advantage, as often as the Power to be reſolved, is much nearer to the Power of the *next greater* whole Number, than to that of the *next leſs*) in this caſe, *Mutatis Mutandis*, we ſhall have the ſame Expreſſions of the Roots, *viz.*

$$\sqrt{aa - b} = \sqrt{aa - b}, \text{ or } a - \frac{ab}{2aa - \frac{1}{2}b}.$$

$$\sqrt[3]{a^3 - b} = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - \frac{b}{3a}}, \text{ or } a - \frac{ab}{3a^3 - b}.$$

$$\sqrt[4]{a^4 - b} = \tfrac{2}{3}a + \sqrt{\tfrac{1}{9}aa - \frac{b}{6aa}}, \text{ or } a - \frac{ab}{4a^4 - \frac{3}{2}b}.$$

$$\sqrt[5]{a^5 - b} = \tfrac{3}{4}a + \sqrt{\tfrac{1}{16}aa - \frac{b}{10a^3}}, \text{ or } a - \frac{ab}{5a^5 - 2b}.$$

$$\sqrt[6]{a^6 - b} = \tfrac{4}{5}a + \sqrt{\tfrac{1}{25}aa - \frac{b}{15a^4}}, \text{ or } a - \frac{ab}{6a^6 - \frac{5}{2}b}.$$

$$\sqrt[7]{a^7 - b} = \tfrac{5}{6}a + \sqrt{\tfrac{1}{36}aa - \frac{b}{21a^5}}, \text{ or } a - \frac{ab}{7a^7 - 3b}.^{*}$$

* The general expreſſions of theſe values of the root of any given number are as follows. Let the given number be called N, and (m being any whole number whatſoever,) let a near value of its mth root that is greater than the truth be called a. Then will the mth root of the given number N be very nearly equal either to the rational expreſſion $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, or to the irrational expreſſion $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$; as is ſhewn in the ſaid large octavo volume of Tracts, in pages 536, 537, - - - 546, where the ſubject is very fully treated, and illuſtrated by examples. F. M.

And

And within theſe two Terms the true Root is ever found, being ſomething nearer to the *Irrational* than the *Rational* Expreſſion. But the quantity e found by the *Irrational Formula* is always too great, as the quotient reſulting from the Rational Formula is always too little. And conſequently, if we have $+ b$, the *Irrational Formula* gives the Root ſomething greater than it ſhould be, and the *Rational* ſomething leſs. But contrary-wiſe, if it be $- b$.

And thus much may ſuffice to be ſaid concerning the extraction of the Roots of pure Powers; which notwithſtanding, for common Uſes, may be had much more eaſily by the help of the Logarithms. But when a Root is to be determin'd very accurately, and the Logarithmick Tables will not reach ſo far, then we muſt neceſſarily have recourſe to theſe, or ſuch like Methods. Farther; the Invention and Contemplation of theſe Formulæ, leading me to a certain Univerſal Rule for Adfected Equations, (which I hope will be of uſe to all the Students in *Algebra* and *Geometry*,) I was willing here to give ſome account of this Diſcovery, which I will do with all the perſpicuity I can. I had given, in No. 188 of the *Tranſactions*, a very eaſy and general conſtruction of all Adfected Equations, not exceeding the Biquadratick Power; from which time I had a very great deſire of doing the ſame in Numbers. But quickly after, Mr. *Raphſon* ſeem'd in great meaſure to have ſatisfy'd this Deſire, till Mr. *de Lagny*, by what he had perform'd in his Book, intimated that the thing might be done more compendiouſly yet. Now, my Method is thus.

Of

Of the Resolution of Affected Equations.

Let z, the root of any Equation, be imagin'd to be compos'd of the parts $a +$ or $- e$, of which let a be assum'd as near z as is possible; which is notwithstanding not *necessary*, but only *commodious*. Then from the Quantity $a + e$ or $a - e$, let there be form'd all the Powers of z, found in the Equation, and let the Numerical Co-efficients of the said Powers in the Terms of the Equation be respectively affixed to them: Then let the Power to be *resolved*, (in the new, or transformed, Equation arising from this substitution of $a + e$, or $a - e$, instead of z, in the Terms of the original Equation,) be subtracted from the sum of the known Parts (in the first Column of Terms in the said new Equation, amongst which the unknown Quantity e is not found) which they call the *Homogeneum Comparationis*, and let the difference be $\pm b$. In the next place, take the sum of all the Co efficients of e in the second Column, to which put $= s$. Lastly, in the third Column let there be put down the sum of all the Co-efficients of ee, which sum call t. Then will the Root z stand thus in the *Rational Formula, viz.* $z = a + \frac{sb}{ss \pm tb}$; and thus in the *Irrational Formula, viz.* $z = a \mp \frac{\frac{1}{2}s \pm \sqrt{\frac{1}{4}ss \mp bt}}{t}$; which perhaps it may be worth while to illustrate by some Examples. And instead of an *Instrument*, let this *Table* serve, which shews the Genesis of the several Powers of $a \pm e$, and if need be, may easily be continued farther; which for its use I may rightly call *A General Analytical Speculum*. The forementioned Powers arising from a continual Multiplication by $a + e$ $(= z)$ come out thus with their adjoyned Co-efficients.

TABELLA

TABELLA POTESTATUM,

or

A GENERAL ANALYTICAL SPECULUM.

$$\begin{array}{llllllll}
 & & s & t & u & w & x & y \\
lz^7 = & la^7 & +7la^6e & +21la^5ee & +35la^4e^3 & +35la^3e^4 & +21la^2e^5 & +7lae^6 + le^7 \\
kz^6 = & ka^6 & +6ka^5e & +15ka^4ee & +20ka^3e^3 & +15ka^2e^4 & +6kae^5 & +ke^6 \\
bz^5 = & ba^5 & +5ba^4e & +10ba^3ee & +10ba^2e^3 & +5bae^4 & +be^5 & \\
gz^4 = & ga^4 & +4ga^3e & +6ga^2ee & +4gae^3 & -ge^4 & & \\
fz^3 = & fa^3 & +3fa^2e & +3faee & +fe^3 & & & \\
dz^2 = & da^2 & +2dae & +dee & & & & \\
cz = & ca & +ce & & & & &
\end{array}$$

But now, if it be $a - e = z$, the Table is compos'd of the fame Members; only the odd Powers of e, as e, e^3, e^5, e^7 are Negative, and the even Powers, as e^2, e^4, e^6, are Affirmative. Alfo let the fum of the Co-efficients of the fide e be $= s$; the fum of the Co-efficients of the Square ee be $= t$, the fum of the Co-efficients of e^3 be $= u$; of $e^4 = w$; of $e^5 = x$; of $e^6 = y$, &c. But now, fince e is fuppos'd to be only a fmall part of the Root that is to be enquir'd, all the Powers of e will be much lefs than the correfpondent Powers of a, and fo, for the firft Hypothefis, all the fuperiour ones may be rejected; and forming a new Equation, by fubftituting $a \pm e = z$, we fhall have (as was faid) $\pm b = \pm se \pm tee$. The following Examples will make this more clear.

Example I.

Let the Equation $z^4 - 3z^2 + 75z = 10{,}000$, be propos'd. For the firft Hypothefis, let $a = 10$, and fo we have this Equation,

$z^4 =$

$$
\begin{array}{rcllll}
z^4 & = & + a^4 & 4a^3e + & 6a^2e^2 & 4ae^3 + e^4 \\
- dz^2 & = & - da^2 & 2dae - & de^2 & \\
+ cz & = & + ca & ce & & \\
& = & + 10000 & 4000e + & 600ee & 40e^3 + e^4 \\
& & - \quad 300 & 60e - & 3ee & \\
& & + \quad 750 & 75e & & \\
& & = 10000 & & & \\
\hline
& & + \quad 450 & 4015e + & 597ee & 40e^3 + e^4 = 0 \\
& & & s & t & u
\end{array}
$$

The Signs + and − with respect to the Quantities e and e^3, are left as doubtful, till it be known whether e be Negative or Affirmative; which thing creates some difficulty, since it sometimes happens, (to wit, in Equations that have several Roots,) that the *Homogeneum Comparationis* (as they term it) is encreased by the diminution of the quantity a, and on the contrary, that, when a is encreased, the said *Homogeneum* is diminished. But the Sign of e is determin'd from the Sign of the Quantity b. For, taking away the *Resolvend* from the *Homogeneum* formed of a, the Sign of se (and consequently of the prevailing Parts in the composition of it) will always be contrary to the Sign of the difference b. Whence 'twill be plain, whether it must be $+ e$, or $- e$; and consequently whether a be taken greater or less than the *True Root*. Now the quantity e is $=$ $\frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, when b and t have the same Sign; but, when the Signs are different, e is $=$ - - $\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$. But after it is found that it will be

be $-e$, let the Powers e, e^3, and e^5, &c. in the Affirmative Members of the Equation be made Negative, and in the Negative be made Affirmative; that is, let them be written with the contrary Signs. On the other hand, if it be $+e$, let those forement,ion'd Powers be made Affirmative in the Affirmative, and Negative in the Negative Members of the Equation.

Now we have in this Example of ours, 10,450 instead of the Resolvend 10,000, or $b = +450$; whence it is plain that a is taken greater than the Truth, and consequently, that 'tis $-e$. Hence the Equation comes to be, $10,450 - 4015e + 597ee - 40e^3 + e^4 = 10,000$. That is, $450 - 4015e + 597ee = 0$; and so $450 = 4015e - 597ee$, or $b = se - tee$; whose Root e is $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, or $\frac{s}{2t} - \sqrt{\frac{ss}{4tt} - \frac{b}{t}}$; that is, in the present case, $e = \frac{2007\frac{1}{2} - \sqrt{3,761,406\frac{1}{4}}}{597}$, from whence we have the Root sought, 9.886, which is near the Truth. But then, substituting this for a second Supposition, there comes $a + e = z$, most accurately 9.886,260,393,649,5 . . . . scarce exceeding the Truth by 2 in the last Figure, *viz.* when $\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$ is $= e$. And this (if need be) may be yet much farther verified, by subtracting (if it be $+e$) the quantity $\frac{\frac{1}{2}ue^3 + \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss + tb}}$, from the Root before found; or (if it be $-e$) by adding $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - tb}}$ to

that

that Root. Which Compendium is so much the more valuable, in that sometimes from the first Supposition alone, but always from the second, a Man may continue the Calculus (keeping the same Co-efficients) as far as he pleases. It may be noted that the fore-mentioned Equation has also a Negative Root, *viz.* $z = 10.26 \ldots.$ which any one, that has a mind, may determine more accurately.

Example II.

Suppose $z^3 - 17z^2 + 54z = 350$, and let $a = 16$. Then according to the prescript of the Rule,

$$\begin{aligned} +z^3 &= a^3 + 3a^2e + 3ae^2 + e^3 \\ -dz^2 &= -da^2 - 2dae - de^2 \\ +cz &= ca + ce \end{aligned}$$

$$\begin{array}{lcccc} & & b & s & t \\ \text{That is,} & +1000 & +300e & +30e^2 & +e^3 \\ & -1700 & -340e & -17e^2 & \\ & +540 & +54e & & \\ & =350 & & & \end{array}$$

Or, $-510 + 14e + 13ee + e^3 = 0$. Now, since we have -510, it is plain, that a is assumed less than the Truth, and consequently that e is Affirmative. And from the Equation $510 = 14e + 13e^2$ comes $e = \dfrac{\sqrt{bt + \frac{1}{4}ss} - \frac{1}{2}s}{t} = \dfrac{\sqrt{6679} - 7}{13} \left(= \dfrac{81.7 - 7}{13} = \dfrac{74.7}{13}\right)$ $= 5.7$. Whence z (or $10 + e$,) is $(= 10 + 5.7) = 15.7\ldots$, which is too much, because of a taken wide. Therefore, Secondly, let $a = 15$, and by the like way of Reasoning, we

we shall find $e = \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - tb}}{t} = \frac{109\frac{1}{2} - \sqrt{11710\frac{1}{4}}}{28}$, and consequently $z = 14.954,068$. If the Operation were to be repeated the third time, the Root will be found conformable to the Truth as far as the 25th Figure; but he that is contented with fewer, by writing $tb \pm te^3$ instead of tb, or subtracting or adding $\frac{\frac{1}{2}e^3}{\sqrt{\frac{1}{4}ss \mp tb}}$ to the Root before found, will presently obtain his end.

Note, the Equation proposed is not explicable by any other Root, because the *Resolvend* 350 is greater than the Cube of $\frac{17}{3}$, or $\frac{d}{3}$.

Example III.

Let us take the Equation $z^4 - 80z^3 + 1998z^2 - 14937z + 5000 = 0$, which Dr. *Wallis* uses, in Cap. 62 of his *Algebra*, in the Resolution of a very difficult Arithmetical Problem, where by *Vieta*'s Method he has obtain'd the root most accurately; and Mr. *Raphson* brings it also as an Example of his Method, in Pages 25, 26. Now this Equation is of the form which may have several Affirmative Roots, and (which increases the difficulty of resolving it) the *Co-efficients* are very great in respect of the *Resolvend* given.

But, that it may be the easier managed, let it be divided, and, according to the known Rules of *Pointing*, let

let $-z^4 + 8z^3 - 20z^2 + 15z = 0,5$ (where the quantity z is $\frac{1}{10}$ of z in the Equation propofed) and for the firft Suppofition, let $a = 1$. Then will $+2 - 5e - 2e^2 + 4e^3 - e^4 - 0,5$ be $= 0$; that is, $1\frac{1}{2}$ will be $= 5e + 2ee$. Here therefore b is $= 1\frac{1}{2}$ or 1.5, and s is $= 5$, and t is $= 2$; and confequently $\frac{1}{4}ss$ is $= \frac{25}{4}$, and bt is $(= 1.5 \times 2 = 3.0) = \frac{12}{4}$, and $\frac{1}{4}ss + bt$ is $= \frac{25}{4} + \frac{12}{4} = \frac{37}{4}$, and $\sqrt{\frac{1}{4}ss + bt}$ is $= \frac{\sqrt{37}}{2}$, and $\sqrt{\frac{1}{4}ss + bt} - \frac{s}{2}$ is $= \frac{\sqrt{37}}{2} - \frac{5}{2} = \frac{\sqrt{37} - 5}{2}$, and

$$\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{s}{2}}{t} \text{ is } = \frac{\frac{\sqrt{37}-5}{2}}{2} = \frac{\sqrt{37} - 5}{4}; \text{ and}$$

hence e $\left(= \frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}\right)$ is $= \frac{\sqrt{37} - 5}{4}$ $\left(= \frac{6.08 - 5}{4} = \frac{1.08}{4}\right) = 0.27$, and fo $a + e$, or z, is $= 1 + 0.27$, or 1.27; whence 'tis manifeft that (10×1.27, or) 12.7 is near the true Root of the Equation propofed. Now, Secondly, let us fuppofe $z = 12.7$, and then, according to the directions of the Table of Powers, there arifes

b	s	t	u	
$-\ 26{,}014.4641$	$-\ 8{,}193.532e$	$-\ 967.74e^2$	$-\ 50.8e^3$	$-\ e^4$
$+\ 163{,}870.640$	$+\ 38{,}709.60e$	$+\ 3048\ e^2$	$+\ 80\ e^3$	
$-\ 322{,}257.42$	$-\ 50{,}749.2\ e$	$-\ 1998\ e^2$		
$+\ 189{,}699.9$	$+\ 14{,}937.\ e$			
$-\ 5000.$				

That

That is, $+ 298.6559 - 5296.132e + 82.26e^2 + 29.2e^3 - e^4 = 0$; And so $- 298.6559 = - 5296.132e + 82.26ee$, whose Root e (being, according to the Rule, $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$) comes to $\frac{2648.066 - \sqrt{6,987,686.106,022}}{82.26} = .056,440,803,31\ldots = e$; which is less than the Truth. But, that it may be corrected, 'tis to be consider'd that $\frac{\frac{1}{2}ue^3 - \frac{1}{2}te^4}{\sqrt{\frac{1}{4}ss - bt}}$, or $\frac{.002,620,1\ldots\ldots}{2643.423\ldots\ldots}$ is .000,000,99, and consequently e corrected is $= 0.056,441,794,48$. And, if you desire yet more Figures of the Root, from the e corrected let there be made $tue^3 - te^4 = 0.431,056,024,23\ldots$, and $\frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt - tue^3 + te^4}}{t}$, or (which is all one,) $\frac{2648.066 - \sqrt{6,987,685.674,965,975,77\ldots\ldots}}{82,26}$, $= .056,441,794,480,744,02 = e$; whence $a + e, = z$ the Root, is most accurately 12.756,441,794,480,744,02... as Dr. *Wallis* found in the fore-mentioned Place; where it may be observ'd, that the repetition of the *Calculus* does ever triple the true Figures in the assumed a, which the first correction, or $\frac{\frac{1}{2}ue^3 - \frac{1}{4}e^4}{\sqrt{\frac{1}{4}ss - bt}}$ does quintuple; which is also commodiously done by the *Logarithms*. But the other Correction, after the first, does also double the number of Figures, so that it renders the *assumed* altogether Seven-fold; yet the first Correction is abundantly sufficient for Arithmetical uses, for the most part.

But,

But, as to what is ſaid concerning the number of Figures rightly taken in the Root, I would have it underſtood ſo that, when a is but $\frac{1}{10}$ part diſtant from the true Root, then the firſt Figure is rightly aſſumed; if it be within $\frac{1}{100}$ part, then the two firſt Figures are rightly aſſumed; and, if within $\frac{1}{1000}$ part, then the three firſt are ſo; which conſequently, managed according to our Rule, do preſently become nine Figures.

It remains now that I add ſomething concerning our *Rational Formula*, viz. $e = \frac{sb}{ss \pm tb}$, which ſeems expeditious enough, and is not much inferiour to the former, ſince it will triple the given Number of Figures. Now, when we ſhall have formed an Equation from $a \pm e = z$, as before, it will preſently appear, whether a be taken greater or leſs than the Truth; ſince se ought always to have a Sign contrary to the Sign of the difference of the *Reſolvend* and its *Homogeneum Comparationis*, produced from a. Then, ſuppoſing $+ b + se + a - tee = o$, the Diviſor is $ss - tb$, as often as t and b have the ſame Signs; but it is $ss + bt$, when they have different ones. But it ſeems moſt commodious for Practice to write the Theorem thus, $e = \cfrac{b}{s \pm \cfrac{tb}{s}}$; ſince in this way the thing is done by one Multiplication and two Diviſions, which otherwiſe would require three Multiplications and one Diviſion.

Let us take now one Example of this Method, from the Root (of the fore-mention'd Equation) 12.7 . . . . , where

298.6559

$$\begin{array}{llll} 298.6559 & -\,5296.132\,e & +\,82.26\,ee & +\,29.2\,e^3 \\ +\,b & -\,s & +\,t & +\,u \end{array}$$

$- e^4$ is $= o$, and so $\dfrac{b}{s - \dfrac{tb}{s}}$ is $= e$; that is, let us

make the following Proportion, to wit, as s is to t, so is b to $\dfrac{tb}{s}$, or as 5296.132 is to 82.26, so is 298.6559 to $\dfrac{tb}{s}$; which will therefore be $\left(= \dfrac{82.26 \times 298.6559}{5296.132}\right)$ $= 4.638,75$; and consequently the Divisor $s - \dfrac{tb}{s}$ will be $(= 5296.132 - 4.638,75) = 5291.493,25$, and $\dfrac{b}{s - \dfrac{tb}{s}}$ will be $\left(= \dfrac{298.6559}{5291.493,25}\right) = 0.056,441$; that is, e will be $= 0.056,441$, in which all the Figures are true. And consequently z, or $a + e$, or the Root of the proposed Equation $z^4 - 80z^3 + 1998z^2 - 14,937z + 5000 = 0$, will be $(= 12.7 + 0.056,441)$ $= 12.756,441$; of which all the eight Figures are true. But this *Rational Formula* $e = \dfrac{sb}{ss \pm tb}$, or $\dfrac{b}{s \pm \dfrac{tb}{s}}$, cannot be corrected, as the foregoing *Irrational* one was; and so, if more Figures of the Root are desired, 'tis the best to make a new Supposition, and repeat the *Calculus* again: And then a new Quotient, tripling the known Figures of the Root, will abundantly satisfie even the most scrupulous.

AN APPENDIX

TO

DR. EDMUND HALLEY'S TRACT ON THE RESOLUTION OF ALGEBRÄICK EQUATIONS OF ALL DEGREES BY APPROXIMATION.

An APPENDIX *to the foregoing Tract of Dr.* EDMUND HALLEY, *intitled,* " *A New, Exact, and Easy, Method of finding the Roots of Equations Generally, and that without any previous Reduction.*"

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Article 1. DR. HALLEY considered his method of resolving affected equations by approximation (which is described in the foregoing tract,) as an improvement upon that which had been given by Mr. Joseph Raphson in his valuable treatise, intitled, *Analysis Æquationum Universalis*, which had been published in the year 1690; because the new figures in the value of the root sought that are discovered by every process of his approximation, were twice as many as the new figures in the said value discovered by the corresponding process of Mr. Raphson's approximation. But this circumstance of quicker dispatch, or rather of the appearance of it, (since the reality of it depends upon the facility and number of the operations to be performed in each

each procefs of the approximation as well as on the number of the faid proceffes,) is not the only thing to be taken into confideration in order to determine which of the two methods is the moft convenient: but the fimplicity and perfpicuity of the reafonings employed in them;—and the greater or lefs danger of falling into miftakes, or difficulties, arifing from having more Algebräick quantities to arrange and manage in Dr. Halley's method than in Mr. Raphfon's;—and fometimes having two roots, or, in the language of modern Algebräifts, two real and affirmative roots, to examine and reafon upon in the proceffes of the former method, and never more than one fuch root in thofe of the latter;—are, all of them, circumftances of great importance in forming a judgement on this queftion. And Mr. Raphfon always continued to prefer his own method of approximation, (which proceeded by the repeated refolutions of mere fimple equations,) to that of Dr. Halley, (which proceeded by the refolution of quadratick equations,) after the publication of the foregoing tract of Dr. Halley as well as before it. For he informs us in the fecond edition of his *Analyfis Æquationum Univerfalis*, (which was publifhed in the year 1697) that he himfelf had had, at the time of preparing that treatife for it's firft publication in the year 1690, the very fame thought as Dr. Halley, of retaining the terms of the fecond, or transformed, equation that involve the fquare of the unknown root of the faid equation as well as the terms that involve the fimple power of the faid root, and fo refolving a quadratick equation inftead of a fimple one; but that, after full confideration of the matter, he had rejected that method

of

of proceeding as liable to more perplexity than the other, and had resolved to proceed only by the resolution of simple equations. And Sir Isaac Newton, in his method of resolving equations by approximation, (which differs a little from Mr. Raphson's,) proceeds only by the resolution of simple equations, except in a few particular cases, in which it may sometimes be convenient to proceed by the resolution of a quadratick equation, as, for example, in performing the first process of approximation to the true value of x, when the first near value of it, obtained by conjecture, or otherwise, and which is taken for the basis of the approximation, is not sufficiently near the truth. And, as far as my experience in these calculations has enabled me to form a judgement on the subject, I am inclined to join with *Mr. Raphson* in preferring his method of proceeding by the resolution of only simple equations, to this of *Dr. Halley* by the resolution of quadraticks. But, that the reader may be the better able to form a judgement for himself upon this subject, I shall now proceed to give a short view of these two different methods of approximating to the values of the roots of equations, and to point out, first, the circumstances in which they agree, and secondly, those in which they differ; and afterwards I shall proceed to consider the three equations $x^4 - 3x^2 + 75x = 10{,}000$, and $x^3 - 17x^2 + 54x = 350$, and $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, (which Dr. Halley has chosen in the foregoing tract for the examples of his method of approximation,) and shall resolve them, first, by Dr. Halley's, and afterwards by Mr. Raphson's method of approximation.

Art. 2.

Art. 2. In both theſe methods of reſolving equations by approximation we muſt begin by finding by conjecture, or otherwiſe, in the beſt manner we can, a firſt near value of x, or the root ſought, which firſt near value Dr. Halley denotes by the letter a. We muſt then ſubſtitute ſuch near value inſtead of x in the compound quantity which forms the left-hand ſide of the propoſed equation, and which contains all the terms that involve the unknown quantity x. Thus, if the propoſed equation is the cubick equation $x^3 + px^2 + qx = r$, and a is found, by conjecture or otherwiſe, to be nearly equal to x, we muſt ſubſtitute a inſtead of x in the compound quantity $x^3 + px^2 + qx$, which will thereby be converted into the compound quantity $a^3 + pa^2 + qa$, which conſiſts intirely of known terms. We muſt then compare this known quantity $a^3 + pa^2 + qa$, reſulting from this ſubſtitution, with r, the abſolute term, or known quantity, of the propoſed equation $x^3 + px^2 + qx = r$: and, if the ſaid reſult is leſs than the ſaid abſolute term, we may conclude that the ſaid near value, a, of the root x of the ſaid equation is leſs than it's true value; and, if the ſaid reſult is greater than the ſaid abſolute term, we may conclude that the ſaid near value is greater than the true value of x. It muſt, however, be obſerved that this concluſion will not always be juſt, becauſe in ſome equations that have more than one root, or in the language of modern Algebräiſts, more than one real and affirmative root, it ſometimes happens that, while the unknown quantity x increaſes, the compound quantity that forms the left-hand ſide of the propoſed equation will decreaſe. Thus, if the propoſed equation had been

x^2

$x^3 - px^2 + qx = r$, inſtead of $x^3 + px^2 + qx = r$, and the magnitudes of the two co-efficients p and q and the abſolute term r had been ſuch that the equation admitted of three real and affirmative roots, and a had been a near value of the middlemoſt of the ſaid three roots, the compound quantity $x^3 - px^2 + qx$ would have been greater than the abſolute term r, if a was leſs than the true value of the middle root of the equation; and, if a was greater than the ſaid true value, the ſaid compound quantity would have been leſs than r. But theſe caſes happen but rarely: and, when they do happen, we muſt make uſe of ſome further reaſonings concerning the roots of the equation and the limits of their magnitudes, in order to determine whether a, or the near value of x which we have choſen for the baſis of our approximation, is greater, or leſs, than it's true value.

Art. 3. When it is determined in the foregoing manner, or by ſome further reaſonings ſuited to the nature of the inquiry, that a is leſs than the true value of x, or the root ſought, we muſt then begin our further approximation to it's true value by ſuppoſing the unknown exceſs of x above a, it's firſt near value, to be denoted by ſome other letter of the alphabet and to be added to a. Dr. Halley on this occaſion makes uſe of the letter e: and we will therefore employ the ſame letter. And then we ſhall have $a + e = x$. We muſt next ſubſtitute this binomial quantity $a + e$ inſtead of x in the propoſed equation. And, if this equation be the cubick equation which was juſt now taken as an example of theſe methods of approximation, to wit, the equation $x^3 + px^2 + qx = r$, we ſhall, by means of this ſubſtitution,

ſubſtitution, convert, or transform, the ſaid equation into the following equation, to wit, $\overline{a+e}|^3 + p \times \overline{a+e}|^2 + q \times \overline{a+e} = r$, or $a^3 + 3a^2e + 3ae^2 + e^3 + p \times \overline{aa + 2ae + ee} + q \times \overline{a+e} = r$, or

$$\left\{\begin{array}{rrrr} a^3 & + 3a^2e & + 3ae^2 & + e^3 \\ + pa^2 & + 2pae & + pe^2 & \\ + qa & + qe & & \end{array}\right\} = r,$$

in which the unknown quantity that remains to be diſcovered is e, or the difference between x and a, inſtead of the original unknown quantity x itſelf. Now this quantity e is leſs than $a + e$, or x, and uſually much leſs than it, namely, only a tenth part of it, or ſome leſs part. And conſequently, as a is nearly equal to x, e will be alſo much leſs than a, and, *à fortiori*, leſs than $3a$, and, ſtill more, leſs than $3a + p$. Therefore e^3 (which is equal to $e \times e^2$) will be much leſs than $\overline{3a+p}| \times e^2$, or than $3ae^2 + pe^2$, and, *à fortiori*, leſs than $\overline{3a^2 + 2pa + q}| \times e$, or than $3a^2e + 2pae + qe$. And conſequently, if e^3 be taken away from the left-hand ſide of the equation, the remaining terms of the ſaid left-hand ſide of the equation, to wit, the terms

$$\left\{\begin{array}{rrr} a^3 & + 3a^2e & + 3ae^2 \\ + pa^2 & + 2pae & + pe^2 \\ + qa & + qe & \end{array}\right\},$$

will ſtill be very nearly equal to the abſolute term r of the firſt equation. But this curtailed equation is only a quadratick equation, and may therefore be reſolved by the

the known methods of refolving fuch equations; which methods are eafy and accurate. Therefore Dr. Halley advifes that, in order to determine the value of e, or the difference between a and x, we fhould reject only the little quantity e^3, or the cube of e, from the foregoing transformed equation, and fhould refolve the remaining quadratick equation by the common method of refolving fuch equations. And for this purpofe he directs us to put s for the compound quantity $3a^2 + 2pa + q$, which is the co-efficient of the unknown quantity e in the faid quadratick equation, and t for the compound quantity $3a + p$, which is the co-efficient of ee, or of the fquare of the unknown quantity e in the fame equation; after which abbreviations of the notation, the equation will be as follows, to wit, $a^3 + pa^2 + qa + se + te^2 = r$. Therefore, (fubtracting $a^3 + pa^2 + qa$ from both fides,) we fhall have $se + te^2 = r - a^3 - pa^2 - qa$, and (dividing both fides by t,) we fhall have $\frac{se}{t} + e^2 = \frac{r - a^3 - pa^2 - qa}{t}$, or (putting b for $r - a^3 - pa^2 - qa$) $\frac{se}{t} + e^2 = \frac{b}{t}$. Therefore $e^2 + \frac{se}{t} + \frac{ss}{4tt}$ will be ($= \frac{b}{t} + \frac{ss}{4tt} = \frac{4bt}{4tt} + \frac{ss}{4tt}$) $= \frac{4bt + ss}{4tt}$, and $e + \frac{s}{2t}$ will be $= \frac{\sqrt{4bt + ss}}{2t}$, and e will be $= \frac{\sqrt{4bt + ss} - s}{2t}$.

Q. E. I.

Therefore

Therefore $a + \frac{\sqrt{4bt + ss} - s}{2t}$ will be a near value of $a + e$, or x, or a second near value of x, which will approach much nearer to it's true value than a, it's first near value, did. And, in general, it will be found that the number of decimal figures in this second near value of x that are exact, or agree with the figures of it's true value, is nearly three times the number of the figures that are exact in a, or it's first near value: or, to express this matter more precisely, it will be found that, if a, the first near value of x, agrees with it's true value in three, or four, or five figures, or, in general, in n figures, the second near value of x, to wit, $a + \frac{\sqrt{4bt + ss} - s}{2t}$, will, accordingly, agree with the said true value in $3 \times 3 - 1$, or $9 - 1$, or 8, figures, or in $4 \times 3 - 1$, or $12 - 1$, or 11, figures, or in $5 \times 3 - 1$, or $15 - 1$, or 14, figures, or, in general, in $3n - 1$ figures, and in the most unfavourable cases in $3n - 2$ figures. This is certainly a great degree of exactness to be attained by a single process of approximation from a, the first near value of x, (obtained by conjecture, or otherwise,) towards it's true value; and it constitutes the great merit of this method of approximation by resolving a quadratick equation, which Dr. Halley has recommended.

Art. 4. But Mr. Raphson has thought proper to reject this method of approximation, notwithstanding this advantage, and to carry on his approximation, from a, the first near value of it, towards $a + e$, or it's true value, by

by refolving only a fimple equation. His method of proceeding is as follows. When he has fubftituted $a + e$ inftead of x in the propofed equation $x^3 + px^2 + qx = r$, and thereby transformed the faid equation into the equation

$$\left\{\begin{array}{l} a^3 + 3a^2e + 3ae^2 + e^3 \\ + pa^2 + 2pae + pe^2 \\ + qa + qe \end{array}\right\} = r,$$

he rejects the two terms $3ae^2$ and pe^2, which involve the fquare of the unknown quantity e, as well as the term e^3, or the cube of the faid unknown quantity, becaufe they are, all of them, much fmaller than the terms $3a^2e + 2pae + qe$, which involve the fimple power of e; and he thereby reduces the faid transformed equation from a cubick equation to a mere fimple equation, to wit, the equation

$$\left\{\begin{array}{l} a^3 + 3a^2e \\ + pa^2 + 2pae \\ + qa + qe \end{array}\right\} = r,$$

or $a^3 + pa^2 + qa + \overline{3a^2 + 2pa + q} \times e = r$; the refolution of which is ftill more fimple and eafy than that of a quadratick equation. For we need only fubtract $a^3 + pa^2 + qa$ from both fides of the equation, by which it becomes $\overline{3a^2 + 2pa + q} \times e = r - a^3 - pa^2 - qa$; or, if (agreeably to Dr. Halley's notation,) we put s for the compound quantity $3a^2 + 2pa + q$ (which is the co-efficient of e,) and b for the abfolute

solute term $r - a^3 - pa^2 - qa$, it will become $se = b$; and consequently e will be $= \frac{b}{s}$. Q. E. I.

Therefore $a + \frac{b}{s}$ will be a near value of $a + e$, or a second near value of x, which will approach much nearer to it's true value than a it's first near value did, but not so near as the former expression $a + \frac{\sqrt{4bt + ss} - s}{2t}$, which was obtained, in the foregoing method recommended by Dr. Halley, by resolving a quadratick equation. For the number of figures in this second near value, $a + \frac{b}{s}$, of the unknown quantity x, that are exact, or agree with the figures of it's true value, will be not nearly triple, but only nearly double, of the number of figures that are exact in a, or it's first near value; or, to express this degree of approximation more precisely, if the number of figures that are exact in a, or the first near value of x, is three, or four, or five, or, in general, n, figures, the number of figures that will be exact in $a + \frac{b}{s}$, or the second near value of x, will, accordingly, be $3 \times 2 - 1$, or $6 - 1$, or 5 figures; or $4 \times 2 - 1$, or $8 - 1$, or 7, figures, or $5 \times 2 - 1$, or $10 - 1$, or 9, figures, or, in general, $2n - 1$ figures, and in the most unfavourable cases will be $2n - 2$ figures. This is a very considerable degree of exactness to be attained by a single process of approximation

approximation performed by the resolution of a mere simple equation, though it is not so great as the degree of exactness attained by means of the former expression $a + \frac{\sqrt{4bt + ss} - s}{2t}$, which is derived from the more difficult operation of resolving a quadratick equation.

Art. 5. And, if the approximation to the value of x by Mr. Raphson's method be carried one step further, by putting c for the quantity $a + \frac{b}{s}$, or the second near value of x, and f for the unknown difference between c and x, and substituting $c - f$ (which will be equal to x,) instead of x in the proposed equation $x^3 + px^2 + qx = r$, and resolving the transformed equation, arising from such substitution, as if it were a mere simple equation, or expunging from it the term f^3, and also the two terms that will involve the square of f, the value of the residual quantity $c - f$ that will be thereby obtained, or the 3d near value of x, will agree with it's true value in nearly twice as many figures as were exact in the 2nd near value c, or $a + \frac{b}{s}$; or, if we put m for the number of figures that were exact in that second near value, (which was $2n - 1$,) the number of figures that will be exact in the value of $c - f$, or the 3d near value of x, will be $2m - 1$, (or $2 \times \overline{2n - 1} - 1$, or $4n - 2 - 1$,) or $4n - 3$, which, if n is any num-

ber greater than 2, will be greater than $3n - 1$, or the number of figures that are exact in the expreſſion $a + \frac{\sqrt{4bt + ss} - s}{2t}$, or the ſecond near value of x obtained by Dr. Halley's method of proceeding. So that, when n, or the number of figures in a, or the firſt near value of x, that are exact, or agree with the figures of it's true value, is greater than 2, the degree of exactneſs that will be attained by two ſteps, or proceſſes, of Mr. Raphſon's method of approximation will be greater than that which is attained by one ſtep, or proceſs, of Dr. Halley's method.

Art. 6. Having now deſcribed theſe two methods of reſolving high equations by approximation which are recommended by theſe two celebrated mathematicians, and having ſhewn how far they co-incide with each other, and in what circumſtances they differ, I ſhall now proceed to illuſtrate the ſubject further in a practical way, by reſolving the three equations which Dr. Halley has choſen in the foregoing Tract as examples of his method of reſolution, to wit, the cubick equation $x^3 - 17xx + 54x = 350$, and the two biquadratick equations $x^4 - 3x^2 + 75x = 10{,}000$ and $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, by both theſe methods, beginning, in each of theſe examples, with the method of Dr. Halley. And, to the end that the reader may be the better able to compare the two methods with each other, and form a judgement of the ſeveral advantages or inconveniencies that belong to each of them, I ſhall,

in

in resolving each of these equations by both methods, begin the approximations to the second near values of them (which have been above denoted by $a + e$,) from the same first near value a, and shall carry the approximations by both methods to nearly the same degree of exactness: and I shall also set forth the grounds and reasons of the conjectures which I shall make concerning the said first near values denoted by a; which has not been done by Dr. Halley in the foregoing Tract: and I shall sometimes make choice of a different first near value a from that which Dr. Halley has adopted, if I shall find that I am able to obtain one that will be nearer to the truth than that which he has chosen, and to support the conjecture to which I give the preference by good and obvious reasons.

EXAMPLES OF THE RESOLUTIONS OF HIGH EQUATIONS BY BOTH DR. HALLEY'S AND MR. RAPHSON'S METHODS OF APPROXIMATION.

EXAMPLE I.—A CUBICK EQUATION.

Art. 7. Let it be required to resolve the cubick equation $x^3 - 17xx + 54x = 350$ by Dr. Halley's method of approximation.

We muſt firſt endeavour to find a tolerably near value of x, which ſhall be true to at leaſt one place of figures, by a conjecture grounded on the obvious properties of this equation. Now ſuch a near value may be found in the manner following.

In this equation $x^3 - 17xx + 54x = 350$ let us, firſt, ſuppoſe x^3 to be equal to $17xx$, and conſequently x to be $= 17$. Then will the compound quantity $x^3 - 17xx + 54x$ be $= 54x = 54 \times 17 = 918$; which is much greater than the abſolute term, 350, of the equation. Therefore the true value of x in this equation muſt be leſs than 17.

We will therefore ſuppoſe, in the ſecond place, that x (which now appears to be conſiderably leſs than 17,) is $= 14$, and ſubſtitute 14 inſtead of it in the compound quantity $x^3 - 17xx + 54x$.

Now, if x be ſuppoſed $= 14$, we ſhall have $xx = 196$, and $x^3 = 2744$, and $17xx$ $(= 17 \times 196) = 3332$, and $54x$ $(= 54 \times 14) = 756$, and conſequently $x^3 - 17xx + 54x$ $(= 2744 - 3332 + 756 = 3500 - 3332) = 168$; which is ſomething leſs than half the abſolute term 350. We will therefore make a third conjecture, and ſuppoſe x to be $= 15$, and try the effect of the ſubſtitution of this number inſtead of x in the compound quantity $x^3 - 17xx + 54x$. Now, if x be ſuppoſed to be $= 15$, we ſhall have $xx = 225$, and $x^3 = 3375$, and $17xx$ $(= 17 \times 225) = 3825$, and

and $54x$ ($= 54 \times 15$) $= 810$, and consequently $x^3 - 17xx + 54x$ ($= 3375 - 3825 + 810 = 4185 - 3825$) $= 360$; which is a little greater than the absolute term 350 of the proposed equation $x^3 - 17xx + 54x = 350$. We may therefore now conclude with certainty that the true value of x in the said equation must be greater than 14 and less than 15; since, while x increases from 14 to 15, the compound quantity $x^3 - 17xx + 54x$ increases from 168 to 360. And the said true value of x will evidently be much nearer to 15 than to 14, because the number 360, to which the said compound quantity is equal when x is equal to 15, approaches much nearer to the number 350, or the absolute term of the proposed equation, than the number 168, which is the value of the said compound quantity when x is $= 14$. We will therefore take the number 15 for a, or the first near value of x in the proposed equation $x^3 - 17xx + 54x = 350$.

Art. 8. Having thus pitched upon the number 15 for a, or the first near value of x, let e be put for the unknown excess of 15, or a, (which we know to be greater than x) above x. Then we shall have $x = a - e$, and x^2 ($= \overline{a - e}|^2$) $= aa - 2ae + ee$, and x^3 ($= \overline{a - e}|^3$) $= a^3 - 3a^2e + 3ae^2 - e^3$, and $17x^2$ ($= 17 \times \overline{aa - 2ae + ee}$) $= 17aa - 34ae + 17ee$, and $54x$ ($= 54 \times \overline{a - e}$) $= 54a - 54e$. Therefore $x^3 - 17xx + 54x$ will be $=$

$$\left\{\begin{array}{l} \quad a^3 - 3a^2e + 3ae^2 - e^3 \\ -17aa + 34ae - 17e^2 \\ +54a - 54e. \end{array}\right\}$$

But $x^3 - 17xx + 54x$ is $= 350$.

Therefore

$$\left\{\begin{array}{l} \quad a^3 - 3a^2e + 3ae^2 - e^3 \\ -17aa + 34ae - 17e^2 \\ +54a - 54e \end{array}\right\}$$

will alſo be $= 350$,

But $a^3 - 17aa + 54a$ has been ſhewn to be $=$ 360; and $3a^2$ is $(= 3 \times \overline{15}|^2 = 3 \times 225) = 675$, and $34a$ is $(= 34 \times 15) = 510$, and $3a$ is $(= 3 \times 15) = 45$.

Therefore we ſhall have $360 - 675e + 510e - 54e + 45e^2 - 17e^2 - e^3 = 350$, or $360 - 219e + 28e^2 - e^3 = 350$, and (adding $219e + e^3$ to both ſides,) $360 + 28e^2 = 350 + 219e + e^3$, and (ſubtracting 350 from both ſides) $10 + 28e^2 = 219e + e^3$, and (ſubtracting $28e^2$ from both ſides,) $219e - 28e^2 + e^3 = 10$, and (omitting e^3 on account of it's ſmallneſs,) $219e - 28e^2$ nearly $= 10$. Therefore (dividing all the terms by 28) we ſhall have $\frac{219e}{28} - e^2 = \frac{10}{28}$, and (ſubtracting both ſides from $\left.\overline{\frac{219}{56}}\right|^2$) we ſhall have $\left.\overline{\frac{219}{56}}\right|^2 - \frac{219e}{28}$

$+ ee$

$+ee = \overline{\frac{219}{56}}|^2 - \frac{10}{28}\ (= \overline{\frac{219}{56}}|^2 - \frac{20}{56} = \frac{\overline{219}|^2}{\overline{56}|^2}$

$- \frac{20 \times 56}{\overline{56}|^2} = \frac{\overline{219}|^2 - 20 \times 56}{\overline{56}|^2} = \frac{47,961 - 1120}{\overline{56}|^2})$

$= \frac{46,841}{\overline{56}|^2}$. Therefore the ſquare-root of the trinomial quantity $\overline{\frac{219}{56}}|^2 - \frac{219e}{28} + ee$ will be $= \frac{\sqrt{46,841}}{56}$ $= \frac{216.427,817, \&c}{56}$. But this trinomial quantity has two ſquare-roots, to wit, $\frac{219}{56} - e$ and $e - \frac{219}{56}$, in the former of which the unknown quantity e is leſs, and in the latter of which it is greater, than $\frac{219}{56}$, or 3.9107. Now the latter of theſe values of e, being greater than 3.9107, will be greater than the value of e which we are ſeeking, or than the exceſs of 15 above x, becauſe x has been ſhewn to be greater than 14, which differs from 15 by only 1. It follows therefore that $\frac{219}{56} - e$, or the former of the two ſquare-roots of the trinomial quantity $\overline{\frac{219}{56}}|^2 - \frac{219e}{28} + e^2$, will be that which is to be uſed on the preſent occaſion. We ſhall therefore have $\frac{219}{56} - e = \frac{216.427,817, \&c}{56}$, and (adding e to both ſides,) $\frac{219}{56} = \frac{216.427,817, \&c}{56} + e$, and (ſubtracting

$\frac{216.427,817, \&c}{56}$ from both sides) e = - - - -

$\frac{219.000,000 - 216.427,817,}{56} = \frac{2.572,183}{56} =$

0.045,931, or, more nearly, 0.045,932. Therefore $a - e$, or $15 - e$, will be = 15.000,000 — 0.045,932 = 14.954,068; that is, the second near value of x, or the root of the proposed equation $x^3 - 17xx + 54x = 350$, obtained by this first process of Dr. Halley's method of resolving equations by approximation, will be 14.954,068. Q. E. I.

Art. 9. This number 14.954,068, here found for the value of x in the proposed equation $x^3 - 17xx + 54x = 350$, is exactly the same as that found by Dr. Halley in the foregoing Tract. And it agrees with the true value of x at least in the first six figures 14.9540. For, if we suppose x to be = 14.9540, we shall have xx = 223.622,116,00, and x^3 = 3344.045,122,664,000, and $17xx$ (= 17 × 223.622,116,00) = 3801.575,972,00, and $54x$ (= 54 × 14.9540) = 807.5160, and consequently $x^3 - 17xx + 54x$ = 3344.045,122,664,000 — 3801.575,972,00 + 807.5160 (= 4151.561,122, 664,000 — 3801.575,972,00) = 349.985,150,664,000; which is somewhat less than the absolute term 350. And, if we suppose x to be = 14.9541, we shall have xx = 223.625,106,81, and x^3 = 3344.112,209,747,421, and $17xx$ (= 17 × 223.625,106,81) = 3801.626,815,77, and $54x$ (= 54 × 14.9541) = 807.5214, and consequently $x^3 - 17xx + 54x$ (= 3344.112,209,747,421 —

$-3801.626,815,77 + 807.5214 = 4151.633,609,747,421 - 3801.626,815,77) = 350.006,793,977,421$; which is somewhat greater than the absolute term 350. Therefore the true value of x must be of an intermediate magnitude between 14.9540 and 14.9541, and consequently the first six figures of it must be 14.9540.

Q. E. D.

The excess of 350.006,793,977,421 above the absolute term 350 is only 0.006,793,977,421, which is less than 0.007; but the excess of the absolute term 350 above 349.985,150,664,000 is 0.014,849,336,000, which is more than double the other excess. It therefore seems reasonable to suppose that the true value of x will approach nearer to 14.9541, which produces the smaller excess, than to 14.9540, which produces the greater excess. And it is very probable that the next two figures 68 of the value 14.954,068, obtained by the foregoing approximation, will likewise be exact; and Dr. Halley seems to have thought they are so. And so they will be found to be in the course of the following resolution of this equation by Mr. Raphson's method of approximation.

The Resolution of the same Equation by Mr. Raphson's Method.

Art. 10. I shall now proceed to resolve the same cubick equation $x^3 - 17xx + 54x = 350$, by Mr. Raphson's method of approximation, beginning the approximation from the same first near value of x, to wit, the number 15, which was chosen in the preceding articles for the basis of the approximation by Dr. Halley's method.

It

It appears from the reasonings used above in art. 7, that the true value of x in this equation is greater than the number 14, and less than the number 15, but is nearer to 15 than to 14. We will therefore take the number 15 for it's first near value a, which is to be the basis of our further approximation to it's true value in the manner prescribed by Mr. Raphson; and we will denote the excess of a, or 15, above the true value of x by the letter e, as we did in the foregoing resolution of the equation by Dr. Halley's method. And we shall then have $x = a - e$, and consequently $xx = \overline{a-e}|^2 = aa - 2ae + ee$, and $x^3 = \overline{a-e}|^3 = a^3 - 3a^2e + 3ae^2 - e^3$, and $17xx$ $(= 17 \times \overline{aa - 2ae + ee}) = 17aa - 34ae + 17ee$, and $54x$ $(= 54 \times \overline{a-e}) = 54a - 54e$. Therefore the compound quantity $x^3 - 17xx + 54x$ will be equal to the compound quantity

$$\left\{\begin{array}{l} \qquad a^3 - 3a^2e + 3ae^2 - e^3 \\ -17a^2 + 34ae - 17e^2 \\ +54a - 54e. \end{array}\right\}$$

But the compound quantity $x^3 - 17xx + 54x$ is $= 350$.

Therefore the compound quantity

$$\left\{\begin{array}{l} \qquad a^3 - 3a^2e + 3ae^2 - e^3 \\ -17a^2 + 34ae - 17e^2 \\ +54a - 54e \end{array}\right\}$$

will also be $= 350$.

This

This equation is accurately true, and is what Mr. Raphſon, in his excellent treatiſe on this ſubject, intitled, *Analyſis Æquationum Univerſalis*, calls *Theorema Vietaeum*, or the Theorem of the great French Algebräiſt *Vieta*, or Monſieur *Viète*, who is often and juſtly ſtyled *The Father of Algebra*, though his works have of late years been but little read. This theorem, or transformed equation ariſing from the ſubſtitution of $a - e$, or $a + e$, inſtead of x in the terms of the original equation $x^3 - 17xx + 54x = 350$, or other propoſed original equation, is the grand foundation of all the methods that have been made uſe of for reſolving equations by approximation, as well as of Vieta's own method ſet forth in his Treatiſe intitled, *De numeroſâ poteſtatum purarum atque adfectarum reſolutione tractatus*, which begins in page 163 and ends in page 228 of Schooten's edition of Vieta's Works publiſhed in the year 1646.

Art. 11. From this equation (which is accurately true) we muſt, according to Mr. Raphſon's method of approximation, expunge the three terms $+ 3ae^2 - 17e^2 - e^3$; and then the equation will be

$$\left\{\begin{array}{r} a^3 - 3a^2e \\ -17a^2 + 34ae \\ +54a - 54e \end{array}\right\} \text{nearly} = 350,$$

or $a^3 - 17a^2 + 54a - 3a^2e + 34ae - 54e =$ (nearly) 350, or (if we ſubſtitute 15 inſtead of a in the terms of this equation) $\overline{15}|^3 - 17 \times \overline{15}|^2 + 54 \times 15 - 3 \times \overline{15}|^2 \times e + 34 \times 15 \times e - 54e = 350$, or

3375

$3375 - 17 \times 225 + 54 \times 15 - 3 \times 225 \times e + 34 \times 15 \times e - 54e = 350$, or $3375 - 3825 + 810 - 675 \times e + 510 \times e - 54e = 350$, or $4185 - 3825 - 729 \times e + 510e = 350$, or $360 - 219e = 350$. Therefore (adding $219e$ to both fides,) we fhall have $360 = 350 + 219e$, and (fubtracting 350 from both fides,) $10 = 219e$, and confequently (dividing both fides by 219) $e = \frac{10}{219} \doteq 0.0456$. Therefore $a - e$, or $15 - e$, will be $= 15.0000 - 0.0456 = 14.9544$; that is, the root x of the propofed equation $x^3 - 17xx + 54x = 350$ will be nearly equal to 14.9544. Q. E. I.

Art. 12. This value of x is exact in the five figures 14.954, which feems to be a great degree of exactnefs to be attained by the refolution of fo eafy a fimple equation as the equation $219e = 10$. And a repetition of this procefs will give it us exact to four or five figures more, or to nine or ten figures in the whole, as will be apparent from the following operations.

Let us fuppofe the firft five figures 14.954 of the foregoing value of x, to wit, 14.9544, to be exact, as we indeed know them to be by means of the fubftitutions made in art. 9; and let us fubftitute this number 14.954 inftead of x in the compound quantity $x^3 - 17xx + 54x$, in order to difcover whether the value of the faid compound quantity arifing from fuch fubftitution will be greater, or lefs, than the abfolute term 350 of the propofed equation, as if we had not already made this in-

quiry

quiry in art. 9, and found the ſaid reſult to be leſs than 350.

Then we ſhall have xx, or $\overline{14.954}|^2$, $= 223.622,116$, and x^3, or $\overline{14.954}|^3$, $= 3344.045,122,664$, and $17xx$ $(= 17 \times 223.622,116) = 3801.575,972$, and $54x$ $(= 54 \times 14.954) = 807.516$, and conſequently $x^3 - 17xx - 54x$ $(= 3344.045,122,664 - 3801.575,972 + 807.516 = 4151.561,122,664 - 3801.575,972) = 349.985,150,664$; which is leſs than 350, or the abſolute term of the propoſed equation $x^3 - 17xx + 54 = 350$. Therefore 14.954 will be leſs than the true value of x in that equation. Let c be put for 14.954, and f for the exceſs of x above 14.954, or c. Then we ſhall have $x = c + f$, and $xx = \overline{c+f}|^2 = cc + 2cf + ff$, and $x^3 = \overline{c+f}|^3 = c^3 + 3c^2f + 3cf^2 + f^3$, and $17xx$ $(= 17 \times \overline{cc + 2cf + ff}) = 17cc + 34cf + 17ff$, and $54x$ $(= 54 \times \overline{c+f}) = 54c + 54f$, and conſequently $x^3 - 17xx + 54x =$

$$\left\{ \begin{array}{l} \qquad\quad c^3 + 3c^2f + 3cf^2 + f^3 \\ -17cc - 34cf - 17f^2 \\ +54c + 54f. \end{array} \right\}$$

But $x^3 - 17xx + 54x$ is $= 350$.

Therefore the compound quantity

$c^3 +$

$$\left\{ \begin{array}{l} c^3 + 3c^2f + 3cf^2 + f^3 \\ -17c^2 - 34cf - 17f^2 \\ +54c + 54f \end{array} \right\}$$

will alſo be $= 350$.

Now let the three terms $+ 3cf^2 - 17f^2 + f^3$ (which will evidently be much ſmaller than the terms $+ 3c^2f - 34cf + 54f$,) be expunged from the equation. And we ſhall then have

$$\left\{ \begin{array}{l} c^3 + 3c^2f \\ -17c^2 - 34cf \\ +54c + 54f \end{array} \right\}$$

$=$ (nearly) 350.

But $c^3 - 17c^2 + 54c$, or $\overline{14.954}^3 - 17 \times \overline{14.954}^2 + 54 \times 14.954$, has been ſhewn to be $= 349.985,150,664$; and $3c^2 \times f$ is $(= 3 \times \overline{14.954}^2 \times f = 3 \times 223.622,116 \times f) = 670.866,348 \times f$, and $34c \times f$ is $(= 34 \times 14.954 \times f) = 508.436 \times f$, and conſequently $3c^2f - 34cf + 54f$ will be $(= 670.866,348 \times f - 508.436 \times f \times 54 \times f = 724.866,348 \times f - 508.436 \times f) = 216.430,348 \times f$.

Therefore $349.985,150,664 + 216.430,348 \times f$ will be $= 350$, and conſequently (ſubtracting $349.985,150,664$ from both ſides,) $216.430,348 \times f$ will be $= 0.014,849,336$, and f will be $= \frac{0.014,849,336}{216.430,348} = 0.000,068,610,2$.

Therefore

Therefore x, or $c+f$, or $14.954+f$, will be $=$ 14.954,068,610,2. Q. E. I.

Art. 13. This number, 14.954,068,610,2, is probably exact in all it's figures; but is certainly so in the first ten figures 14.954,068,61, of which the first eight figures, 14.954,068, are the same as those of the former value of x, obtained by Dr. Halley's method of approximation. So that one process of Dr. Halley's method of approximation gives us the value of x in the cubick equation $x^3 - 17xx + 54x = 350$ exact to eight figures by the resolution of the quadratick equation $219e - 28e^2 = 10$, and two processes of Mr. Raphson's method of approximation give us the value of the same quantity exact to at least ten places of figures by the resolution of the two simple equations $219e = 10$, and $216.430,348 \times f = 0.014,849,336$. The reader must now judge for himself which of these two methods deserves the preference.

EXAMPLE II.—A BIQUADRATICK EQUATION.

Art. 14. Let it be required to resolve the biquadratick equation $x^4 - 3x^2 + 75x = 10,000$ by Dr. Halley's method of approximation.

An Investigation, by Conjectures and Trials, of a, *or the first near Value of* x.

In order to find a first near value of x in this equation, to be the basis of a second approximation to it's true

value

value in the method recommended by Dr. Halley, we may proceed as follows:

If x^4 alone were equal to the absolute term 10,000, x would be exactly equal to 10. Therefore it seems reasonable to conjecture that in the equation $x^4 - 3xx + 75x = 10,000$, (in which the other two terms $3xx$ and $75x$ have contrary signs prefixed to them, and therefore have less effect in increasing or diminishing the magnitude of x^4 than they would have if they were both marked with the same sign, and consequently will not make the magnitude of the trinomial quantity $x^4 - 3xx + 75x$ be very different from that of x^4 alone,) the value of x will not be very different from 10. We will therefore substitute 10 instead of x in the compound quantity $x^4 - 3xx + 75x$, in order to discover whether the value of it resulting from this supposition will be nearly equal to the absolute term 10,000. Now, if x be supposed to be $= 10$, we shall have $xx = 100$, and and $x^4 = 10,000$, and $3xx = 300$, and $75x = 750$, and consequently $x^4 - 3xx + 75x$ $(= 10,000 - 300 + 750) = 10,450$; which is greater than the absolute term 10,000, but not in any great degree. We will therefore make a second supposition not very different from the former, to wit, that x is equal to 9, and try the effect of this supposition. Now, if x is $= 9$, we shall have $xx = 81$, and $x^4 = 6561$, and $3xx = 243$, and $75x$ $(= 75 \times 9) = 675$, and consequently $x^4 - 3xx + 75x$ $(= 6561 - 243 + 675 = 7236 - 243) = 6993$. Therefore, while x increases from 9 to 10, the compound quantity $x^4 - 3xx + 75x$ will increase from 6993 to 10,450, and therefore will, at some one

point

point of time during the said increase of x from 9 to 10 be equal to 10,000, or the absolute term of the proposed equation $x^4 - 3xx + 75x = 10,000$; or there will be some quantity greater than 9, but less than 10, which, being substituted instead of x in the compound quantity $x^4 - 3xx + 75x$, will make the said quantity be equal to 10,000; that is, in other words, the true value of x in the equation $x^4 - 3xx + 75x = 10,000$ will be greater than 9, but less than 10. And, as the value of the said compound quantity $x^4 - 3xx + 75x$ resulting from the substitution of 10 instead of x in it's terms was 10,450, which exceeds 10,000 by only 450, and the value of the said compound quantity resulting from the substitution of 9 instead of x in it's terms was only 6993, which falls short of 10,000 by the number 3007, which is a much greater quantity than 450, it seems reasonable to conclude that 10 will differ less than 9 from the true value of x in the said equation $x^4 - 3xx + 75x = 10,000$. And for these reasons we will pitch upon 10 for the value of a, or the first near value of x in the proposed equation, or for the basis of the approximation which we are now to make towards it's true value in the manner recommended by Dr. Halley.

A more exact Determination of the Value of x *by Dr. Halley's Method of Approximation.*

Art. 15. Now let e be put for the difference between a, or 10, and the true value of x in the proposed equation.

And we shall then have $x = 10 - e$, or $a - e$, and consequently $xx = \overline{a-e}|^2 = aa - 2ae + ee$, and

x^3

$x^4 = \overline{a - e}|^4 = a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4$, and $3xx$ ($= 3 \times \overline{aa - 2ae + ee}$) $= 3a^2 - 6ae + 3ee$, and $75x$ ($= 75 \times \overline{a - e}$) $= 75a - 75e$. Therefore the whole compound quantity $x^4 - 3xx + 75x$ will be equal to the compound quantity

$$\left\{\begin{array}{l} \quad a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ - 3a^2 + 6ae - 3e^2 \\ + 75a - 75e. \end{array}\right\}$$

But the compound quantity $x^4 - 3xx + 75x$ is $= 10,000$.

Therefore the compound quantity

$$\left\{\begin{array}{l} \quad a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ - 3a^2 + 6ae - 3e^2 \\ + 75a - 75e \end{array}\right\}$$

will alfo be $= 10,000$.

This is the full transformed equation, and is accurately true. But from this equation Dr. Halley directs us to expunge the two terms $4ae^3$ and e^4, as being very fmall in comparifon of the terms $6a^2e^2$ and $3e^2$, and, *à fortiori*, in comparifon of the terms $4a^3e$, $6ae$, and $75e$, which are to be retained. And by this omiffion the equation is reduced to the quadratick equation

$$\left\{\begin{array}{l} \quad a^4 - 4a^3e + 6a^2e^2 \\ - 3a^2 + 6ae - 3e^2 \\ + 75a - 75e \end{array}\right\} \text{ nearly} = 10,000.$$

Art. 16.

Art. 16. Now, ſince a is $= 10$, we ſhall have $a^4 - 3a^2 + 75a = 10,450$, as has been ſhewn in art. 14; and we ſhall have $4a^3$ $(= 4 \times \overline{10}|^3 = 4 \times 1000) = 4000$, and $6a$ $(= 6 \times 10) = 60$, and $6a^2$ $(= 6 \times \overline{10}|^2 = 6 \times 100) = 600$, and conſequently

$$\left\{\begin{array}{l} a^4 - 4a^3e + 6a^2e^2 \\ -3a^2 + 6ae - 3e^2 \\ +75a - 75e \end{array}\right\} =$$

$$\left\{\begin{array}{rrr} 10,450 & -\ 4000e & +\ 600e^2 \\ & +\ \ \ 60e & -\ \ \ 3e^2 \\ & -\ \ \ 75e & \end{array}\right\}$$

$$= 10,450 - 4015e + 597e^2.$$

Therefore $10,450 - 4015e + 597e^2$ will be nearly $= 10,000$. And conſequently (adding $4015e$ to both ſides,) we ſhall have $10,450 + 597e^2 = 10,000 + 4015e$, and (ſubtracting $597e^2$ from both ſides,) $10,450 = 10,000 + 4015e - 597e^2$, and, laſtly, (ſubtracting 10,000 from both ſides,) $4015e - 597e^2 = 450$. This quadratick equation muſt now be reſolved; which may be done as follows.

Art. 17. Divide all the terms by 597, the co-efficient of e^2. And we ſhall have $6.725,293 \times e - e^2 = 0.753,769$. Now let both ſides of this equation be ſubtracted from the ſquare of half the number 6.725,293, which is the co-efficient of e, that is, from the ſquare of 3.362,646, or from 11.307,388,121,316; and we ſhall

then

then have $\overline{3.362,646}^2 - 6.725,293 \times e + e^2$ (= 11.307,388,121,316 − 0.753,769) = 10.553,619,121, 316. Therefore the square-root of the trinomial quantity $\overline{3.362,646}^2 - 6.725,293 \times e + e^2$ will be (= $\sqrt{10.553,619,121,316}$) = 3.248,633.

But this trinomial quantity has two square-roots, to wit, the binomial quantity $3.362,646 - e$, and the binomial quantity $e - 3.362,646$, in which latter quantity e is greater than 3.362,646. But we have seen above in art. 14, that e, or the difference between 10 and the true value of x, is less than 1. Therefore this latter binomial quantity $e - 3.362,646$ cannot be that which will give us the true value of e, and consequently we must make use of the other square-root of the said trinomial quantity, to wit, the binomial quantity $3.362,646 - e$. We shall therefore have $3.362,646 - e = 3.248,633$; and consequently (adding e to both sides,) we shall have $3.362,646 = 3.248,633 + e$, and e (= 3.362,646 − 3.248,633) = 0.114,013, or (neglecting the three last places of figures as not exact,) 0.114. Therefore $a - e$, or $10 - e$, will be = 10 − 0.114 = 9.886; that is, the second near value of x in the proposed equation $x^4 - 3xx + 75x = 10,000$ will be 9.886.

Q. E. I.

Art. 18. Now let 9.886 be substituted instead of x in the compound quantity $x^4 - 3xx + 75x$, in order to discover whether the value of that quantity resulting from

from such substitution will be greater, or less, than the absolute term 10,000, and consequently whether the said number 9.886 will be greater, or less, than the true value of x in the proposed equation $x^4 - 3xx + 75x = 10,000$. This substitution may be made as follows.

If x is $= 9.886$, we shall have xx $(= \overline{9.886}|^2) = 97.732,996$, and x^3 $(= \overline{9.886}|^3) = 966.188,398,456$, and x^4 $(= \overline{9.886}|^4) = 9551.738,507,136,016$, and consequently $3xx$ $(= 3 \times 97.732,996) = 293.198,988$, and $75x$ $(= 75 \times 9.886) = 741.450$, and the whole compound quantity $x^4 - 3xx + 75x$ $(= 9551.738, 507,136,016 - 293.198,988 + 741.450 = 10,293.188, 507,136,016 - 293.198,988) = 9,999.989,519,136,016$; which is very nearly equal to, but somewhat less than, the absolute term, 10,000, of the proposed equation $x^4 - 3xx + 75x = 10,000$. Therefore the number 9.886 will be very nearly equal to, but somewhat less than, the true value of x in that equation. Q. E. I.

A second Process of Dr. Halley's Method of Approximation.

Art. 19. Now, in order to obtain the value of x to a greater degree of exactness by means of a second process of Dr. Halley's method of approximation, let c be put for the value of x already found, to wit, 9.886, and f for the unknown difference by which the true value of x exceeds 9.886.

Then we shall have $x = c + f$, and consequently xx $(= \overline{c+f}|^2) = cc + 2cf + ff$, and x^4 $(= \overline{c+f}|^4) = c^4 + 4c^3f + 6c^2f^2 + 4cf^3 + f^4$, and $3xx$ (=

3 ×

$3 \times \overline{cc + 2cf + ff}) = 3cc + 6cf + 3ff$, and $75x$ $(= 75 \times \overline{c+f}) = 75c + 75f$. Therefore the compound quantity $x^4 - 3xx + 75x$ will be equal to the compound quantity

$$\left\{\begin{array}{l} \quad c^4 + 4c^3f + 6c^2f^2 + 4cf^3 + f^4 \\ -3cc - 6cf \quad -3f^2 \\ +75c + 75f. \end{array}\right\}$$

Therefore this last compound quantity will be = the absolute term 10,000.

From this equation let the two terms $4cf^3$ and f^4 be expunged, on account of their extreme smallness in comparison of the terms that involve ff and f. And we shall then have the compound quantity

$$\left\{\begin{array}{l} \quad c^4 + 4c^3f + 6c^2f^2 \\ -3c^2 - 6cf - 3f^2 \\ +75c + 75f \end{array}\right\}, \text{ nearly, } = 10{,}000.$$

But we have seen that $c^4 - 3c^2 + 75c$, or $\overline{9.886})^4 - 3 \times \overline{9.886})^2 + 75 \times 9.886$, is $= 9{,}999.989{,}519{,}136{,}016$; and $4c^3$ will be $(= 4 \times \overline{9.886})^3 = 4 \times 966.188{,}398{,}456) = 3864.753{,}593{,}824$, and $6c$ will be $(= 6 \times 9.886) = 59.316$, and consequently $4c^3 - 6c + 75$ will be $(= 3864.753{,}593{,}824 - 59.316 + 75 = 3939.753{,}593{,}824 - 59.316)$ $3880.437{,}593{,}824$; and $6c^2$ will be $(= 6 \times \overline{9.886})^2 = 6 \times 97.732{,}996)$ $= 586.397{,}976$, and consequently $6c^2 - 3$ will be $(= 586.397{,}976 - 3) = 583.397{,}976$. Therefore the compound quantity

c^4

$$\left\{\begin{array}{l} c^4 + 4c^3f + 6c^2f^2 \\ -3c^2 - 6cf - 3f^2 \\ +75c + 75f \end{array}\right\}$$

will be $= 9,999.989,519,136,016 + 3880.437,593,824 \times f + 583.397,976 \times ff$. And consequently this laſt quantity $9,999.989,519,136,016 + 3880.437,593,824 \times f + 583.397,976 \times ff$ will be, nearly, $= 10,000$, and therefore (ſubtracting 9,999.989,519,136,016 from both ſides,) $3880.437,593,824 \times f + 583.397,976 \times ff$ will be $= 0.010,480,863,984$; which is a quadratick equation now properly reduced into order, and prepared for reſolution. And by reſolving this equation we ſhall obtain the value of f, and conſequently of $c + f$, or $9.686 + f$, or the third near value of x in the propoſed equation $x^4 - 3xx + 75x = 10,000$. This quadratick equation may be reſolved as follows.

Art. 20. Since $3880.437,593,824 \times f + 583.397,976 \times ff$ is $= 0.010,480,863,984$, we ſhall have

$\dfrac{3880.437,593,824 \times f}{583.397,976} + ff = \dfrac{0.010,480,863,984}{583.397,976}$, that is, $6.651,441,646 \times f + ff = 0.000,017,965,204,569,033$. Now let the ſquare of half the co-efficient of f, that is, of half the number 6.651,441,646, or the ſquare of 3.325,720,823, (which ſquare is $= 11.060,418,992,535,797,329$,) be added to both ſides of this equation. And we ſhall have $\overline{3.325,720,823}|^2 + 6.651,441,646 \times f + ff$ $(= 11.060,418,992,535,797,329 + 0.000,017,965,204,569,033) = 11.060,436,957,740,366,362$. Therefore (extracting the ſquare-roots of both ſides,) we ſhall have $3.325,720,823 + f = 3.325,723,523$; and con-

consequently f will be ($= 3.325,723,523 - 3.325,720,823$) $= 0.000,002,700$. Therefore $c + f$, or $9.886 + f$, will be ($= 9.886 + 0.000,002,700$) $= 9.886,002,700$; that is, the third near value of x in the proposed equation $x^4 - 3xx + 75x = 10,000$, obtained by Dr. Halley's method of approximation, will be 9.886,002,700. Q. E. I.

N. B. Dr. Halley (in the foregoing tract, as it is printed in the *Miscellanea Curiosa*,) makes this more accurate value of x to be 9.886,260,393,649,5, which, he tells us, scarce exceeds the truth by 2 in the last figure (see above, page 15.) But I conceive that, if this is not owing to some errors of the press, he must have made some slip in his calculation; because the number here found for x, to wit, 9.886,002,700, (with which Dr. Halley's number agrees in only the first four figures 9.886,) will be found to agree in the first nine figures 9.886,002,70, with the number that will be found for it in the following articles by Mr. Raphson's method of approximation.

The Resolution of the same Equation by Mr. Raphson's Method of Approximation.

Art. 21. I now proceed to resolve the same biquadratick equation $x^4 - 3xx + 75x = 10,000$ by Mr. Raphson's method of approximation, beginning the approximation from the same first near value of x, to wit, the number 10, which was chosen in the foregoing articles for the basis of the approximation by Dr. Halley's method.

Having found, by the reasonings used in art. 14, that the value of x in the proposed equation $x^4 - 3xx + 75x = 10,000$ will be nearly equal to, but somewhat less

lefs than, the number 10, let a be $= 10$, and let e denote the unknown difference by which 10 exceeds the true value of x.

Then will x be $= 10 - e$, or $a - e$; and confequently xx will be $(= \overline{a-e}|^2) = aa - 2ae + ee$, and x^3 will be $(= \overline{a-e}|^3) = a^3 - 3a^2e + 3ae^2 - e^3$, and x^4 will be $(= \overline{a-e}|^4) = a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4$. Therefore $3xx$ will be $= 3 \times \overline{aa - 2ae + ee}| = 3aa - 6ae + 3ee$, and $75x$ will be $(= 75 \times \overline{a-e}) = 75a - 75e$. Therefore the compound quantity $x^4 - 3xx + 75x$ will be $=$ the compound quantity

$$\left\{\begin{array}{l} \quad a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ - 3a^2 + 6ae \ - 3e^2 \\ + 75a - 75e. \end{array}\right\}$$

Therefore this laft compound quantity will be $=$ the abfolute term 10,000.

Now let the four terms $+ 6a^2e^2 - 3e^2 - 4ae^3 + e^4$ be expunged from this equation, as being much fmaller than the three preceeding terms $4a^3e$, $6ae$, and $75e$, which involve the fimple power of e. And the remaining compound quantity

$$\left\{\begin{array}{l} \quad a^4 - 4a^3e \\ - 3a^2 + 6ae \\ + 75a - 75e \end{array}\right\}$$

will be nearly $=$ the abfolute term 10,000.

But $a^4 - 3a^2 + 75a$ has been fhewn in art. 14 to be $= 10,450$; and $4a^3$ is $(= 4 \times \overline{10}|^3 = 4 \times 1000) = 4000$, and $6a$ is $(= 6 \times 10) = 60$.

Therefore

Therefore $a^4 - 3a^2 + 75a - 4a^3e + 6ae - 75e$ will be $(= 10{,}450 - 4000e + 60e - 75e = 10{,}450 - 4075e + 60e) = 10{,}450 - 4015e$; and consequently $10{,}450 - 4015e$ will be, nearly, = the absolute term 10,000. Therefore, (adding $4015e$ to both sides,) we shall have $10{,}000 + 4015e = 10{,}450$, and (subtracting 10,000 from both sides,) $4015e = 450$. Therefore e will be $= \frac{450}{4015} = 0.112$. Therefore $a - e$, or $10 - e$, will be $= 10 - 0.112 = 9.888$; that is, the second near value of x, or the root of the proposed equation $x^4 - 3xx + 75x = 10{,}000$, will be 9.888.

Q. E. I.

Art. 22. Of this number, 9.888, the first three figures, 9.88, are exact, or the same with the three first figures of the true value of x, and the fourth figure 8 is only a little greater than the truth, the exact fourth figure being a 6. And the resolution of the simple equation $4015e = 450$, by which this value of e has been obtained, is much easier and shorter than the resolution of the quadratick equation $4015e - 597ee = 450$, given above in art. 17, by which we obtained the number 9.886 for the second near value of x, by Dr. Halley's method.

Art. 23. Having thus obtained 9.888 for a second near value of x in the proposed equation $x^4 - 3xx + 75x = 10{,}000$, we will try the degree of it's approximation to the truth by substituting it instead of x in the compound quantity $x^4 - 3xx + 75x$.

Now, if x is $= 9.888$, we shall have xx $(= \overline{9.888}|^2) = 97.772{,}544$, and x^3 $(= \overline{9.888}|^3) = 966.774{,}915{,}072$, and

and x^4 (= $\overline{9.888}|^4$) = 9559.470,360,231,936, and consequently $3xx$ (= 3 × 97.772,544) = 293.317,632, and $75x$ (= 75 × 9.888) = 741.600, and $x^4 + 75x$ (= 9559.470,360,231,936 + 741.600) = 10,301.070,360,231,936, and $x^4 - 3xx + 75x$ (= 10,301.070,360,231,936 — 293.317,632) = 10,007.752,728,231,936; which is a little greater than 10,000, or the absolute term of the equation $x^4 - 3xx + 75x = 10{,}000$. Therefore 9.888 will be a little greater than the true value of x in that equation.

A second Process of Mr. Raphson's Method of Approximation.

Art. 24. In order to obtain a nearer value of x, let f be put for the excess of 9.888 above the true value of x. And we shall then have $x = 9.888 - f$, and consequently xx (= $\overline{9.888 - f}|^2 = \overline{9.888}|^2 - 2 \times 9.888 \times f +$ &c) = $97.772{,}544 - 19.776 \times f +$ &c, and x^4 (= $\overline{9.888 - f}|^4 = \overline{9.888}|^4 - 4 \times \overline{9.888}|^3 \times f +$ &c = $9559.470{,}360{,}231{,}936 - 4 \times 966.774{,}915{,}072 \times f +$ &c) = $9559.470{,}360{,}231{,}936 - 3867.099{,}660{,}288 \times f +$ &c, and $3xx$ (= $3 \times \overline{97.772{,}544 - 19.776 \times f + \&c} = 3 \times 97.772{,}544 - 3 \times 19.776 \times f$) = $293.317{,}632 - 59.328 \times f +$ &c, and $75x$ (= $75 \times \overline{9.888 - f} = 75 \times 9.888 - 75 \times f$) = $741.600 - 75f$.

Therefore the compound quantity $x^4 - 3xx + 75x$ will be = the compound quantity

9559.

$$\left\{\begin{array}{lll} 9559.470,360,231,936 & - \; 3867.099,660,288 & \times f + \&c \\ -293.317,632, & + \quad 59.328 & \times f - \&c \\ +741.600 & - \quad 75 \times f & \end{array}\right\}$$

$= 10,007.752,728,231,936 - 3882.771,660,288 \times f$ &c.

But the compound quantity $x^4 - 3xx + 75x$ is $=$ 10,000.

Therefore we ſhall have $10,007.752,728,231,936 - 3882.771,660,288 \times f$ &c alſo $= 10,000$; and, (adding $3882.771,660,288 \times f$ to both ſides,) $10,007.752,728,231,936$ nearly $= 10,000 + 3882.771,660,288 \times f$, and (ſubtracting 10,000 from both ſides,) $3882.771,660,288 \times f =$ nearly, $7.752,728,231,936$,

and $f = \dfrac{7.752,728,231,936}{3882.771,660,288} = 0.001,99$.

Therefore $9.888 - f$ will be $= 9.888 - 0.001,99 = 9.886,01$; that is, the third near value of x in the propoſed equation $x^4 - 3xx + 75x = 10,000$ will be $9.886,01$. Q. E. I.

Art. 25. If x is $= 9.886,01$, we ſhall have

$xx \; (= \overline{9.886,01}|^2) = 97.733,193,720,1$,

and $x^3 \; (= \overline{9.886,01}|^3) = 966.191,330,448,845,801$,

and $x^4 \; (= \overline{9.886,01}|^4) = 9551.777,154,730,594,077,144,01$,

and conſequently $3xx \; (= 3 \times 97.733,193,720,1) = 293.199,581,160,3$

and $75x \; (= 75 \times 9.886,01) = 741.450,75$,

and

and $x^4 + 75x = 10,293.227,904,730,594,077,144,01$, and $x^4 + 75x - 3xx$, or $x^4 - 3xx + 75x$, $=$

$$10,293.227,904,730,594,077,144,01$$
$$-\ 293.199,591,160,3$$

$= 10,000.028,313,570,294,077,144,01$; which is a little greater than 10,000, or the abfolute term of the propofed equation $x^4 - 3xx + 75x = 10,000$. And confequently 9.886,01 will be a little greater than the true value of x in that equation.

A third Procefs of Mr. Raphfon's Method of Approximation.

Art. 26. In order to obtain a ftill nearer value of x, let g be put for the excefs of 9.886,01 above it's true value. And we fhall then have $x = 9.886,01 - g$, and xx $(= \overline{9.886,01 - g}|^2 = \overline{9.886,01}|^2$

$$- 2 \times 9.886,01 \times g + \&c) =$$
$$97.733,193,720,1 - 19.772,02 \times g + \&c$$

and x^4 $(= \overline{9.886,01 - g}|^4 = \overline{9.886,01}|^4 - 4 \times$

$$\overline{9.886,01}|^3 \times g + \&c$$
$$= \overline{9.886,01}|^4 - 4 \times 966.191,330,448,845,801 \times g + \&c$$
$$= \overline{9.886,01}|^4 - 3864.765,321,795,383,204 \times g + \&c)$$
$$= 9551.777,154,730,594,077,144,01$$
$$- 3864.765,321,795,383,204 \times g + \&c,$$

and $3xx$ $(= 3 \times \overline{97.733,193,720,1 - 19.772,02 \times g + \&c}$

$$= 3 \times 97.733,193,720,1 - 3 \times 19.772,02 \times g + \&c)$$
$$= 293.199,591,160,3 - 59.316,06 \times g + \&c,$$

and $75x$ $(= 75 \times \overline{9.886,01 - g} = 75 \times 9.886,01 - 75g)$ $= 741.450,75 - 75g$, and confequently $x^4 - 3xx + 75x =$ the compound quantity

9551.

$$\left\{\begin{array}{ll} 9551.777.154,730,594,077,144,01 & - 3864.765,321,795,383,204 \times g + \&c \\ - 293.199,591,160,3 & + \ 59.316,06 \times g - \&c \\ + 741.450,75 & - \ 75 \times g \end{array}\right\}$$

$$= 10,000.028,313,570,294,077,144,01$$
$$- 3880.449,261,795,383,204 \times g + \&c.$$

Therefore this last quantity 10,000.028,313,570,294, 077,144,01 — 3880.449,261,795,383,204 $\times g$ + &c will be nearly = 10,000.

Therefore (adding 3880.449,261,795,383,204 $\times g$ to both sides,) we shall have 10,000.028,313,570,294,077, 144,01 = 10,000 + 3880.449,261,795,383,204 $\times g$, and (subtracting 10,000 from both sides,) 3880.449,261,795, 383,204 $\times g$ = 0.028,313,570,294,077,144,01, and, lastly, (dividing both sides by 3880.449,261,795,383,204) g = 0.000,007,296,467.

Therefore x, or 9.886,01 — g, will be = 9.886,01 — 0.000,007,296,467 = 9.886,002,703,533; that is, the fourth near value of x in the proposed equation $x^4 - 3xx + 75x = 10,000$, obtained by this third process of Mr. Raphson's method of approximation, will be 9.886,002,703,533. Q. E. I.

Art. 27. Of this number 9.886,002,703,533, the first nine figures, 9.886,002,70, are the same with the first nine figures of the last value of x found above in art. 20 by Dr. Halley's method of approximation, to wit, 9.886, 002,700. And therefore we may be confident that these nine figures, 9.886,002,70 are exact, or the same with the first nine figures of the true value of x in the proposed

posed equation $x^4 - 3xx + 75x = 10{,}000$. And I believe that the first eleven figures, to wit, 9.886,002, 703,5, of the value obtained in the foregoing article, to wit, 9.886,002,703,533, are exact.

Dr. Halley makes the more accurate value of x in this equation, which he has obtained by the second process of his method of approximation, to be 9.886,260,393,649,5, (see above, page 15.) which, he tells us, scarce exceeds the truth by 2 in the last figure. But from the agreement of the two values of x found in the preceeding articles by the application of both his and Mr. Raphson's methods of approximation, in the first nine figures 9.886,002,70, we may safely conclude that these nine figures are exact, and consequently that the value of x assigned by Dr. Halley is erroneous in all the figures 260,393,649,5 that come after the four first figures 9.886.

I now proceed to consider Dr. Halley's third and last example.

EXAMPLE III.—A BIQUADRATICK EQUATION.

Art. 28. Let it be required to resolve the biquadratick equation $-x^4 + 80x^3 - 1998x^2 + 14{,}937x = 5000$, or $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, by Dr. Halley's method of approximation.

An Investigation, by Conjectures and Trials, of a, *or the first near Value of* x.

In order to find a first near value of one of the roots of this biquadratick equation (for it has four real and affirmative roots,) to be made the basis of a second approximation to it's true value in the manner recommended by Dr. Halley, we may proceed as follows:

Let

Let us fubftitute two or three very eafy numbers, confifting of two figures each, inftead of x, in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, which forms the left-hand fide of the propofed equation, in order to fee whether either of the values refulting from fuch fubftitutions will be nearly equal to 5000, or the abfolute term of the faid equation. And, firft, let us fuppofe x to be equal to 10.

Then will the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ be $= 14,937 \times 10 - 1998 \times 100 + 80 \times 1000 - 10,000$ $(= 149,370 - 199,800 + 80,000 - 10000 = 229,370 - 209,800) = 19,570$; which is confequently greater than the abfolute term 5000. We will, therefore, in the fecond place, fuppofe x to be $= 12$.

Then we fhall have $14,937x - 1998x^2 + 80x^3 - x^4 = 14,937 \times 12 - 1998 \times 144 + 80 \times 1728 - 20,736$ $(= 179,244 - 287,712 + 138,240 - 20,736 = 317,484 - 308,448) = 9,036$; which is lefs than the former refult, 19,570, but yet is greater than the abfolute term 5000.

We will therefore, in the third place, fuppofe x to be $= 13$.

And then we fhall have $14,937x - 1998x^2 + 80x^3 - x^4 = 14,937 \times 13 - 1998 \times 169 + 80 \times 2197 - 28,561$ $(= 194,181 - 337,662 + 175,760 - 28,561 = 369,941 - 366,223) = 3,718$; which is lefs than the abfolute term 5000.

Therefore, if we fuppofe x to increafe from 12 to 13, the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will have decreafed at the fame time from 9,036 to 3,718, and therefore muft, at fome inftant of time during

during the ſaid decreaſe, have been equal to the intermediate quantity 5000; or, in other words, there will be a quantity greater than 12, but leſs than 13, that will be a root of the propoſed equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$.

Further, ſince the reſult of the ſubſtitution of 12 inſtead of x in the compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ is 9,036, and the reſult of the ſubſtitution of 13 inſtead of x in the ſame quantity is 3,718, which differs leſs than the former reſult, or 9,036, from the abſolute term 5000, it ſeems reaſonable to ſuppoſe that the true value of x in the equation $14,937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$ will approach nearer to 13 than to 12; and it alſo ſeems reaſonable to ſuppoſe that the difference of the two extreme numbers 12 and 13, correſponding to the reſults 9036 and 3718, will be to the difference of 12 and x in nearly the ſame proportion as the difference of the ſaid reſults 9036 and 3718 to the difference of the former reſult 9036, (which correſponds to 12) and the abſolute term 5000, or that $13 - 12$ will be to $x - 12$ in nearly the ſame proportion as $9036 - 3718$ is to $9036 - 5000$, or that 1 will be to $x - 12$ in nearly the ſame proportion as 5318 is to 4036, and conſequently that $x - 12$ will be nearly equal to $\frac{1 \times 4036}{5318}$, or to 0.7, or that x will be nearly equal to $12 + 0.7$, or 12.7. And therefore we will take 12.7 for our firſt near value of x, which we denote by the letter a, and will make it the baſis of our further approximation to the true value of x in the propoſed equation in the method recommended by Dr. Halley.

A more exact Determination of the Value of x *by Dr. Halley's Method of Approximation.*

Art. 29. As we do not yet know whether 12.7 is greater, or less, than x, let 12.7 be substituted instead of x in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to discover whether the result will be greater, or less, than the absolute term 5000, and from that circumstance to determine whether 12.7 is greater or less than x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Now, if x is $= 12.7$, we shall have xx $(= \overline{12.7}|^2) = 161.29$, and x^3 $(= \overline{12.7}|^3) = 2048.383$, and x^4 $(= \overline{12.7}|^4) = 26,014.4641$, and consequently $14,937x$ $(= 14,937 \times 12.7) = 189,699.9$, and $1998x^2$ $(= 1998 \times 161.29) = 322,257.42$, and $80x^3$ $(= 80 \times 2048.383) = 163,870.640$, and the whole quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ $(= 189,699.9 - 322,257.42 + 163,870.640 - 26,014.4641 = 353,570.5400 - 348,271.8841) = 5298.6559$; which is somewhat greater than 5000, or the absolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore, while x increases from 12 to 12.7, the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will have decreased only from 9036 to 5298.6559. But we have seen that, when x is $= 13$, the said compound quantity will be $= 3718$. Therefore, while x increases from 12.7 to 13, the said compound quantity will decrease from 5298.6559 to 3718, and consequently will, at some instant of time during it's said decrease, be equal to the intermediate quantity 5000, or the absolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$;

5000; and therefore there will be ſome value of x greater than 12.7, but leſs than 13, that, being ſubſtituted inſtead of x in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, will make the ſaid compound quantity be equal to 5000; or, in other words, there will be a root of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ that will be greater than 12.7, but leſs than 13.

Art. 30. Having thus diſcovered that the true value of that root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ which we are now ſeeking, is greater than 12.7, let e be put for it's unknown exceſs above 12.7, ſo that x ſhall be $= 12.7 + e$, or $a + e$.

Then we ſhall have

$$xx\ (= \overline{a+e}|^2) = aa + 2ae + ee,$$

and $x^3\ (= \overline{a+e}|^3) = a^3 + 3a^2e + 3ae^2 + e^3$,

and $x^4\ (= \overline{a+e}|^4) = a^4 + 4a^3e + 6a^2e^2 + 4ae^3 + e^4$,

and conſequently

$14,937x\ (= 14937 \times \overline{a+e}) = 14,937a + 14,937e$,

and $1998x^2\ (= 1998 \times \overline{aa + 2ae + ee}) =$

$$1998aa + 3996ae + 1998ee,$$

and $80x^3\ (= 80 \times \overline{a^3 + 3a^2e + 3ae^2 + e^3}) =$

$$80a^3 + 240a^2e + 240ae^2 + 80e^3;$$

and conſequently the whole quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be equal to the following compound quantity, to wit,

$$\left\{\begin{array}{lllll} & 14,937a & + 14,937e & & \\ - & 1998aa & - 3996ae & - 1998ee & \\ + & 80a^3 & + 240a^2e & + 240ae^2 & + 80e^3 \\ - & a^4 & - 4a^3e & - 6a^2e^2 & - 4ae^3 - e^4. \end{array}\right\}$$

But

But the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the last-mentioned compound quantity, involving the unknown quantity e instead of x, will also be $= 5000$.

Now let the three last terms of this last compound quantity, to wit, the terms $+ 80e^3 - 4ae^3 - e^4$, be expunged from it, on account of their smallness in comparison of the terms in the second and third vertical columns of the said compound quantity, which involve in them the square and the simple power of the unknown quantity e. And we shall then have the remaining compound quantity, to wit, the quantity

$$\left\{\begin{array}{rrrrrr} & 14,937a & + & 14,937e & & \\ - & 1998a^2 & - & 3996ae & - & 1998e^2 \\ + & 80a^3 & + & 240a^2e & + & 240ae^2 \\ - & a^4 & - & 4a^3e & - & 6a^2e^2 \end{array}\right\}.$$

nearly $= 5000$.

Now the first vertical column of terms in this equation, to wit, $14937a - 1998a^2 + 80a^3 - a^4$, has already been shewn (in art. 29,) to be $= 5298.6559$; and $3996 \times a$ is $(= 3996 \times 12.7) = 50,749.2$; and $240a^2$ is $(= 240 \times \overline{12.7}|^2 = 240 \times 161.29) = 38,709.60$; and $4a^3$ is $(= 4 \times \overline{12.7}|^3 = 4 \times 2048.383) = 8193.532$; and $240a$ is $(= 240 \times 12.7) = 3048.0$; and $6a^2$ is $(= 6 \times \overline{12.7}|^2 = 6 \times 161.29) = 967.74$.

Therefore we shall have the compound quantity

$$\left\{\begin{array}{llll} 5298.6559 + 14.937\,e & & & \\ & -\ 50,749.2 \times e & -\ 1998 & \times e^2 \\ & +\ 38,709.60 \times e & +\ 3048.0 & \times e^3 \\ & -\ \ 8,193.532 \times e & -\ \ 967.74 & \times e^2 \end{array}\right\}$$

nearly = 5000, or the compound quantity

$$\left\{\begin{array}{l} 5298.6559 + 53,646.60 \times e - 2965.74 \times e^2 \\ \qquad\qquad - 58,942.732 \times e + 3048.0 \times e^2 \end{array}\right\}$$

= 5000, or the compound quantity $5298.6559 - 5296.132 \times e + 82.26 \times e^2 = 5000$. Therefore (adding $5296.132 \times e$ to both ſides,) we ſhall have $5298.6559 + 82.26 \times e^2 = 5000 + 5296.132 \times e$; and (ſubtracting $82.26 \times e^2$ from both ſides,) we ſhall have $5298.6559 = 5000 + 5296.132 \times e - 82.26 \times e^2$; and, laſtly, (ſubtracting 5000 from both ſides,) we ſhall have $5296.132 \times e - 82.26 \times e^2 = 298.6559$; which is a quadratick equation properly prepared for reſolution, and which we will therefore now proceed to reſolve, in order to obtain the value of e, or of the exceſs of the true value of x in the propoſed equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$ above a, or 12.7, it's firſt near value.

Art. 31. Let all the terms of this equation $5296.132 \times e - 82.26 \times e^2 = 298.6559$ be divided by 82.26, the co-efficient of e^2. And we ſhall then have $64.382,834,9 \times e - e^2 = 3.630,633,357,646,48$. Now let both ſides of this equation be ſubtracted from the ſquare of half the co-efficient of e, that is, from the ſquare of half of 64.382,834,9, or the ſquare of 32.191,417,4, or from 1036.287,354,221,022,76. And we ſhall have $\overline{32.191,417,4}|^2 - 64.382,834,9 \times e + e^2$ (= 1036.287,

354,

354,221,022,76 − 3.630,633,357,646,48) = 1032.656, 720,863,376,28. Therefore the ſquare-root of the trinomial quantity $\overline{32.191,417,4}|^2 - 64.382,834,9 \times e + e^2$ will be equal to the ſquare-root of 1032.656,720, 863,376,28, that is, to 32.134,976,5.

But the trinomial quantity $\overline{32.191,417,4}|^2 - 64.382,834,9 \times e + e^2$ has two ſquare-roots, to wit, the binomial quantity 32.191,417,4 — e and the binomial quantity e — 32.191,417,4. We muſt therefore now inquire which of theſe two ſquare-roots will enable us to find the value of e required on the preſent occaſion, or that value of it which is equal to the exceſs of the root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ above 12.7. Now we know that this exceſs is leſs than 0.3, becauſe x is leſs than 13. Therefore the latter ſquare-root of the ſaid trinomial quantity, to wit, the binomial quantity e − 32.191,417,4, (in which e is greater than 32.191,417,4.) cannot be that which is ſuited to our preſent purpoſe; and conſequently we muſt make uſe of the other ſquare-root of the ſaid trinomial quantity, to wit, the binomial quantity 32.191, 417,4 — e. We ſhall therefore have 32.191,417,4 − e = 32.134,976,5; and conſequently (adding e to both ſides,) 32.191,417,4 will be = 32.134,976,5 + e, and e will be (= 32.191,417,4 − 32.134,976,5) = 0.056,440,9. Therefore $a + e$, or 12.7 + e, will be (= 12.7 + 0.056,440,9) = 12.756,440,9; that is, the ſecond near value of x, or the root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, obtained by this firſt proceſs of Dr. Halley's method of approximation, will be 12.756,440,9. Q. E. I.

This value of x is exact in the first seven figures 12.756,44, it's more accurate value, as computed by both Dr. Halley and Dr. Wallis, being 12.756,441,794,480,744,02.

A Correction of the foregoing, or second, near Value of x, *suggested by Dr. Halley.*

Art. 32. Dr. Halley on this occasion points out a correction to be made to the value of x just now obtained, to wit, the number 12.756,440,9, without entering upon a compleat second process of his method of approximation, and tells us that we may, by this correction, find the value of x to be = 12.756,441,794,48, or to thirteen figures, all exact. This correction, if I understand it, may be explained as follows.

The compleat transformed equation obtained in art. 30 by substituting $a + e$ instead of x in the terms of the original equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, is

$$\left\{\begin{array}{lllll} & 14,937a & + 14,937e & & \\ - & 1998a^2 & - 3996ae & - 1998e^2 & \\ + & 80a^3 & + 240a^2e & + 240ae^2 & + 80e^3 \\ - & a^4 & - 4a^3e & - 6a^2e^2 & - 4ae^3 - e^4 \end{array}\right\}$$

$= 5000$.

And by substituting 12.7, or the value of a, instead of a in all the terms of this equation, except $80e^3 - 4ae^3 - e^4$, this equation will be converted into the following equation, to wit, $5298.6559 - 5296.132 \times e + 82.26 \times e^2 + 80e^3 - 4ae^3 - e^4 = 5000$.

Add $5296.132 \times e$ to both sides: and we shall have $5298.6559 + 82.26 \times e^2 + 80e^3 - 4ae^3 - e^4 = 5000 + 5296.132 \times e$.

Subtract

Subtract $82.26 \times e^2$ from both ſides: and we ſhall have $5298.6559 + 80e^3 - 4ae^3 - e^4 = 5000 + 5296.132 \times e - 82.26 \times e^2$.

Laſtly, ſubtract 5000 from both ſides: and we ſhall have $5296.132 \times e - 82.26 \times e^2 = 298.6559 + 80e^3 - 4ae^3 - e^4$, or $5296.132 \times e - 82.26 \times e^2 = 298.6559 + 80e^3 - 4 \times 12.7 \times e^3 - e^4$, or $5296.132 \times e - 82.26 \times e^2 = 298.6559 + 80e^3 - 50.8 \times e^3 - e^4$, or $5296.132 \times e - 82.26 \times e^2 = 298.6559 + 29.2 \times e^3 - e^4$. Therefore (dividing all the terms by 82.26, (the co-efficient of e^2,) we ſhall have $64.382,834,913,688 \times e - e^2 = 3.630,633,357,646,486,749,331,388 + 0.354,972,039,873, \times e^3 - \frac{e^4}{82.26}$.

Now let the value of e that we have already found, to wit, 0.05644, be inſerted inſtead of e in the two terms $0.354,972,039,873 \times e^3$ and $\frac{e^4}{82.26}$; and for this purpoſe let e, or 0.05644, be raiſed to it's third and fourth powers.

Then we ſhall have $e^2 = 0.003,185,473,6$, and $e^3 = 0.000,179,788,129,984$, and $e^4 = 0.000,010,147,242,056,296,96$. Therefore $0.354,972,039,873 \times e^3$ will be $(= 0.354,972,039,873 \times 0.000,179,788,129,984) = 0.000,063,819,759,245,372,554,852,032$; and $\frac{e^4}{82.26}$ will be $= \frac{0.000,010,147,242,056,296,96}{82.26} = 0.000,000,$

123,

123,355,726,432,007,780; and consequently 0.354,972, 039,873 $\times e^3 - \frac{e^4}{82.26}$ will be (= 0.000,063,819,759, 245,372,554,852,032 — 0.000,000,123,355,726,432, 007,780) = 0.000,063,696,403,518,940,547,072,032; and 3.630,633,357,646,486,749,331,388 + 0.354,972, 039,873 $\times e^3 - \frac{e^4}{82.26}$ will be (= 3.630,633,357, 646,486,749,331,388 + 0.000,063,696,403,518,940,547, 072,032) = 3.630,697,054,050,005,689,878,460,032. Therefore 64.382,834,913,688 $\times e - e^2$ will be = 3.630,697,054,050,005,689,878,460,032; which quadratick equation we must now resolve.

Art. 33. The half of 64.382,834,913,688 (the co-efficient of e) is 32.191,417,456,844, the square of which is 1036.287,357,880,800,614,602,440,336. From this number let both sides of the equation 64.382,834,913, 688 $\times e - e^2$ = 3.630,697,054,050,005,689,878,460,032 be subtracted. And we shall have the trinomial quantity $\overline{32.191,417,456,844}|^2$ — 64.382,834,913,688 $\times e + e^2$ (= 1036.287,357,880,800,614,602,440,336 — 3.630, 697,054,050,005,689,878,460,032) = 1032.656,660, 826,750,608,912,561,875,968. Therefore (extracting the square-roots of both sides,) we shall have 32.191,417, 456,844 — e = 32.134,975,662,457; and consequently (adding e to both sides,) we shall have 32.191,417,456, 844 = 32.134,975,662,457 + e, and (subtracting 32.134,975,662,459 from both sides) e = 0.056,441, 794,387. Therefore $a + e$, or 12.7 + e will be (= 12.7 + 0.056,441,794,387) = 12.756,441,794,387; that is, the value of x in the proposed equation 14,937 x —

1998x^2

$1998x^2 + 80x^3 - x^4 = 5000$ will be 12.756,441, 794,387. Q. E. I.

A Remark on the foregoing Correction of the second near Value of x, *suggested by Dr. Halley.*

Art. 34. This number 12.756,441,794,387 is exact in the first eleven figures 12.756,441,794, whereas the former number, obtained in art. 31, to wit, 12.756,440,9, was exact in only the first seven figures 12.756,44; so that this correction has given us four new figures of the value of x exact. And Dr. Halley makes the number obtained by means of the correction exact to two figures more, it being according to his calculation 12.756,441, 794,48, which consists of thirteen figures, which are all exact. But even this additional number of exact figures seems to be hardly worth the trouble of going through the calculations that are necessary to obtain them. If more figures of the value of x are required than are afforded by the first process of Dr. Halley's method of approximation, I should rather be inclined to enter upon a compleat second process of it, grounded upon the value found by the first process, than to have recourse to this correction of the first process; partly because I believe the labour of calculation in performing a compleat second process will hardly be greater than that of applying the foregoing correction to the first process, and partly, and chiefly, because I think the reasonings employed in a second process are clearer and more satisfactory than those employed in making the correction of the first process, and less likely to lead the calculator into a mistake, such as that of adding a quantity where it should be subtracted, or the contrary, or of omitting some necessary division of the

the new quantities taken into the calculation, or not carrying the division far enough. But the comparative merits of a compleat second process of Dr. Halley's method of approximation to the root of an equation, and of the foregoing correction of the first process of it, will be better understood by exhibiting a compleat second process of it in the case of the foregoing equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, grounded on the value of x obtained by the first process, which was 12.756,440,9, or rather on 12.756,44, the first seven figures of that value, because those figures will be sufficient to enable us to find the next value of x exact to about 21 places of figures; and therefore I shall now proceed to exhibit the said second process in the same full manner as the former process.

An Investigation of a third near Value of x *by a second Process of Dr. Halley's Method of Approximation.*

Art. 35. We will therefore, in the first place, substitute the number 12.756,44 instead of x in the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$, in order to discover whether the value of that quantity resulting from such substitution will be greater, or less, than 5000, or the absolute term of the proposed equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, and from that circumstance to determine whether the number 12.756,44 is greater, or less, than the true value of x in that equation.

Now, if x is $=$ 12.756,44, we shall have

$$xx\ (= \overline{12.756{,}44}|^2) = 162.726{,}761{,}473{,}6;$$

and $$x^3\ (= \overline{12.756{,}44}|^3) = 2075.814{,}169{,}132{,}289{,}984;$$

and

and x^4 $(= \overline{12.756,44}^{\,4}) = 26,479.998,899,685,909,243,496,96$;

and consequently $14,937\,x$ $(= 14,937 \times 12.756,44) = 190,542.944,28$;

and $1998\,xx$ $(= 1998 \times 162.726,761,173,6) = 325,128.069,424,252,8$;

and $80\,x^3$ $(= 80 \times 2075.814,169,132,289,984) = 166,065.133,530,583,198,720$.

Therefore the whole compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ will be (=

$$\left\{ \begin{array}{r} 190,542.944,28 - 325,128.069,424,252,8 \\ + 166,065.133,530,583,198,720 - 26,479.998,899,685, \\ 909,243,496,96 \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 356,608.077,810,583,198,720 \\ -351,608.068,323,938,709,243,496,96 \end{array} \right\}$$

$= 5000.009,486,644,489,476,503,04$; which is a very little greater than 5000, or the absolute term of the proposed equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$. Therefore, for the reasons given above in art. 29, we may conclude that the number 12.75644 is less than the true value of x in that equation.

Art. 36. Having thus discovered that the number 12.75644 is less than the true value of x in the equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, let us put c for the number 12.75644, and f for the unknown excess of x above it, so that x will be equal to $c + f$.

Then

Then we ſhall have $xx\ (=\overline{c+f}|^2) = cc + 2cf + ff$, and $x^3\ (=\overline{c+f}|^3) = c^3 + 3c^2f + 3cf^2 + f^3$, and $x^4\ (=\overline{c+f}|^4) = c^4 + 4c^3f + 6c^2f^2 + 4cf^3 + f^4$, and $14{,}937\,x\ (= 14{,}937 \times \overline{c+f}) = 14{,}937\,c + 14{,}937\,f$, and $1998\,x^2\ (= 1998 \times \overline{cc + 2cf + ff}) = 1998\,cc + 3996\,cf + 1998\,ff$, and $80\,x^3\ (= 80 \times \overline{c^3 + 3c^2f + 3cf^2 + f^3}) = 80\,c^3 + 240\,c^2f + 240\,cf^2 + 80\,f^3$. And conſequently the whole compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$ will be equal to the compound quantity

$$\left\{\begin{array}{llllll} & 14{,}937\,c & + 14{,}937\,f & & & \\ - & 1998\,cc & - 3996\,cf & - 1998\,ff & & \\ + & 80\,c^3 & + 240\,c^2f & + 240\,cff & + 80\,f^3 & \\ - & c^4 & - 4\,c^3f & - 6\,c^2ff & - 4\,cf^3 & - f^4. \end{array}\right\}$$

But the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$ is $= 5000$.

Therefore the other compound quantity, which involves in it's terms the unknown quantity f inſtead of x, will alſo be equal to 5000.

Now let the three laſt terms, $+ 80\,f^3 - 4\,cf^3 - f^4$, be expunged from the ſaid compound quantity, on account of their extream ſmallneſs in compariſon to the preceding terms which involve the ſquare and the ſimple power of f. And the remaining compound quantity, to wit,

14,937

$$\left\{\begin{array}{llll} 14{,}937\,c & + 14{,}937\,f & & \\ - 1998\,cc & - 3996\,cf & - 1998\,f^2 & \\ + \quad 80\,c^3 & + 240\,c^2 f & + 240\,cf^2 & \\ - \quad c^4 & - \quad 4\,c^3 f & - 6\,c^2 f^2 & \end{array}\right\}$$

will be, nearly, $= 5000$.

But it has been ſhewn in art. 35 that the firſt vertical column of terms in this equation, to wit, $14{,}937\,c - 1998\,cc + 80\,c^3 - c^4$ is $= 5000.009{,}486{,}644{,}489{,}476{,}503{,}04$; and $3996\,c$ is $(= 3996 \times 12.75644) = 50{,}974.734{,}24$;
and $240\,c^2$ is $(= 240 \times 162.726{,}761{,}473{,}6)$
$= 39{,}054.422{,}753{,}664{,}0$;
and $4\,c^3$ is $(= 4 \times 2075.814{,}169{,}132{,}289{,}984)$
$= 8303.256{,}676{,}529{,}159{,}936$;
and $240\,c$ is $(= 240 \times 12.756{,}44) = 3061.545{,}60$;
and $6\,c^2$ is $(= 6 \times 162.726{,}761{,}473{,}6)$
$= 976.360{,}568{,}841{,}6$.

Therefore we ſhall have the compound quantity

$$\left\{\begin{array}{l} 5000.009{,}486{,}644{,}489{,}476{,}503{,}04 \\ \qquad + 14{,}937\,f \\ \qquad - 50{,}974.734{,}24 \times f \\ \qquad + 39{,}054.422{,}753{,}664{,}0 \times f \\ \qquad - 8{,}303.256{,}676{,}529{,}159{,}936 \times f \\ \qquad\qquad - 1998 \times f^2 \\ \qquad\qquad + 3061.545{,}60 \times f^2 \\ \qquad\qquad - 976.360{,}568{,}841{,}6 \times f^2 \end{array}\right\}$$

nearly $= 5000$, or the compound quantity

5000.

$$\left\{\begin{array}{l} 5000.009,486,644,489,476,503,04 \\ \quad + 53,991.422,753,664,0 \times f \\ \quad - 59,277.990,916,529,159,936 \times f \\ \quad - 2974.360,568,841,6 \times f^2 \\ \quad + 3061.545,60 \times f^2 \end{array}\right\}$$

nearly = 5000, or the compound quantity

$$\begin{array}{l} 5000.009,486,644,489,476,503,04 \\ \quad - 5286.568,162,865,159,936 \times f \\ \quad + 87.184,431,158,4 \times f^2, \end{array}$$

nearly, = 5000; and consequently (adding 5,286.568,162,865,159,936 $\times f$ to both sides,) 5000.009,486,644,489,476,503,04 + 87.184,431,158,4 $\times f^2$ = 5000 + 5286.568,162,865, 159,936 $\times f$, and (subtracting 87.184,431,158,4 $\times f^2$ from both sides,) 5000.009,486,644,489,476,503,04 = 5000 + 5286.568,162,865,159,936 $\times f$ — 87.184, 431,158,4 $\times f^2$, and, lastly, (subtracting 5000 from both sides,) 5286.568,162,865,159,936 $\times f$ — 87.184, 431,158,4 $\times f^2$ = 0.009,486,644,489,476,503,04; which is a quadratick equation properly prepared for resolution, and which we will therefore now proceed to resolve in order to obtain the value of f, or of the excess of the true value of x in the proposed equation 14,937 x — 1998 x + 80 x^3 — x^4 = 5000 above c, or 12.75644, it's second near value.

Art. 37. Let all the terms of this equation 5286.568, 162,865,159,936 $\times f$ — 87.184,431,158,4 $\times f^2$ = 0.009,486,644,489,476,503,04 be divided by 87.184,431, 158,4, which is the co-efficient of f^2. And we shall then have 60.636,607,851,008,642,270,08 $\times f$ — f^2

=

and $1998x^2$ will be $(= 1998 \times \overline{aa + 2ae + ee})$
$= 1998aa + 3996ae + 1998ee$,

and $80x^3$ will be $(= 80 \times \overline{a^3 + 3a^2e + 3ae^2 + e^3})$
$= 80a^3 + 240a^2e + 240ae^2 + 80e^3$,

and confequently the whole compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be equal to the compound quantity

$$\left\{\begin{array}{rrrrr} & 14,937a & + 14,937e & & \\ - & 1998a^2 & - 3996ae & - 1998e^2 & \\ + & 80a^3 & + 240a^2e & + 240ae^2 & + 80e^3 \\ - & a^4 & - 4a^3e & - 6a^2e^2 & - 4ae^3 - e^4. \end{array}\right\}$$

But the former compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the latter compound quantity will alfo be $= 5000$.

Now let all the fix terms of the faid latter compound quantity which involve in them the fquare, cube, and fourth power, of the unknown quantity e, to wit, the terms $- 1998e^2 + 240ae^2 - 6a^2e^2 + 80e^3 - 4ae^3 - e^4$, be expunged from the faid compound quantity on account of their fmallnefs in comparifon of the terms $+ 14,937e - 3996ae + 240a^2e - 4a^3e$, which involve only the fimple power of e: and it is evident that the remaining compound quantity, to wit, the compound quantity $14,937a - 1998a^2 + 80a^3 - a^4 + 14,937e - 3996ae + 240a^2e - 4a^3e$, will be, nearly, $= 5000$.

But

But it has been ſhewn in art. 29 that, if a be equal to 12.7, (as it is here ſuppoſed to be,) the quadrinomial quantity $14,937\,a - 1998\,a^2 + 80\,a^3 - a^4$ will be $= 5298.6559$.

Therefore the compound quantity $5298.6559 + 14,937\,e - 3996\,ae + 240\,a^2e - 4\,a^3e$ will be nearly equal to 5000.

Further, ſince a is $= 12.7$, we ſhall have $a^2 = 161.29$, and $a^3 = 2048.383$, as is ſhewn in art. 29. Therefore $3996\,ae$ will be $(= 3996 \times 12.7 \times e) = 50,749.2 \times e$, and $240\,a^2e$ will be $(= 240 \times 161.29 \times e) = 38,709.60 \times e$, and $4\,a^3 \times e$ will be $(= 4 \times 2048.383 \times e) = 8193.532 \times e$.

Therefore the compound quantity $5298.6559 + 14,937\,e - 50,749.2 \times e + 38,709.60 \times e - 8193.532 \times e$ will be nearly $= 5000$, or the compound quantity $5298.6559 + 53,646.60 \times e - 58,942.732 \times e$ will be nearly $= 5000$, or the compound quantity $5298.6559 - 5296.132 \times e$ will be nearly $= 5000$. Therefore (adding $5296.132 \times e$ to both ſides,) we ſhall have $5298.6559 = 5000 + 5296.132 \times e$, and (ſubtracting 5000 from both ſides,) we ſhall have $5296.132 \times e = 298.6559$; which is a ſimple equation properly prepared for reſolution.

This ſimple equation $5296.132 \times e = 298.6559$ is reſolved by the ſingle operation of dividing both ſides of

the equation by the number 5296.132, which is the coefficient of the unknown quantity e. And by this division we shall find e to be $(= \frac{298.6559}{5296.132}) = 0.056,39$. Therefore $a + e$, or $12.7 + e$, will be $(= 12.7 + 0.056,39) = 12.756,39$; that is, the second near value of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, obtained by this first process of Mr. Raphson's method of approximation, will be 12.756,39. Q. E. I.

Art. 39. Now let 12.75639 be substituted instead of x in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to discover whether the result of such substitution will be greater, or less, than 5000, or the absolute term of the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and from that circumstance to determine whether the true value of x will be greater or less than 12.75639. This substitution will be as follows.

If x is $= 12.756,39$, we shall have

xx $(= \overline{12.75639}|^2) = 162.725,485,832,1$,

and x^3 $(= \overline{12.75639}|^3) = 2075.789,760,213,742,119$,

and x^4 $(= \overline{12.75639}|^4) = 26,479.583,749,292,977,829,390,41$.

Therefore $14,937x$ will be $(= 14,937 \times 12.756,39) = 190,542.197,43$,

and $1998x^2$ will be $(= 1998 \times 162.725,485,832,1) = 325,125.520,692,535,8$,

and

and $80x^3$ will be ($= 80 \times 2075.789,760,243,742,119$)
$= 166,063.180,817,099,369,520$,

and consequently the whole compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be ($= 190,542.197,43 - 325,135.520,692,535,8 + 166,063.180,817,099,369,520 - 26,479.583,749,292,977,829,390,41 = 356,605.378,247,099,369,520 - 351,605.104,441,828,777,829,390,41$) $= 5000.273,805,270,591,690,609,59$; which is somewhat greater than 5000, or the absolute term of the proposed equation. Therefore, while x increases from 12.7 to 12.756,39, the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will have decreased from 5298.6559 to 5000.273,805,270,591,690,609,59; and, while x increases farther from 12.75639 to 13, the said compound quantity will have decreased farther from 5000.273,805,270,591,690,609,59, to 3718. Therefore there will be some value of x greater than 12.75639, but less than 13, that will make the said compound quantity be equal to the intermediate quantity 5000; or, in other words, the true value of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be greater than 12.756,39, but less than 13.

Q. E. I.

A second Process of Mr. Raphson's Method of Approximation.

Art. 40. Now let c be put $= 12.756,39$, and f for the unknown quantity by which the true value of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ exceeds c, or 12.756,39; so that x shall be $= c + f$, or $12.756,39 + f$.

Then

Then we ſhall have $xx\ (=\overline{c+f}|^2) = cc + 2cf + ff$, and $x^3\ (=\overline{c+f}|^3) = c^3 + 3c^2f + 3cf^2 + f^3$, and $x^4\ (=\overline{c+f}|^4) = c^4 + 4c^3f + 6c^2f^2 + 4cf^3 + f^4$, and conſequently

$14{,}937x\ (= 14{,}937 \times \overline{c+f}) = 14{,}937 \times c + 14{,}937 \times f$,

and $1998\,xx\ (= 1998 \times \overline{cc + 2cf + ff})\ 1998\,cc + 3996\,cf + 1998\,ff$,

and $80\,x^3\ (= 80 \times \overline{c^3 + 3c^2f + 3cf^2 + f^3}) = 80\,c^3 + 240\,c^2f + 240\,cf^2 + 80\,f^3$,

and the whole compound quantity $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4$ = the compound quantity

$$\left\{\begin{array}{llllll} & 14{,}937\,c & + 14{,}937\,f & & & \\ - & 1998\,cc & - 3996\,cf & - 1998\,ff & & \\ + & 80\,c^3 & + 240\,c^2f & + 240\,cff & + 80\,f^3 & \\ - & c^4 & - 4\,c^3f & - 6\,c^2ff & - 4cf^3 & - f^4. \end{array}\right\}$$

But the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$ is $= 5000$.

Therefore the compound quantity

$$\left\{\begin{array}{llllll} & 14{,}937\,c & + 14{,}937\,f & & & \\ - & 1998\,cc & - 3996\,cf & - 1998\,ff & & \\ + & 80\,c^3 & + 240\,c^2f & + 240\,cff & + 80\,f^3 & \\ - & c^4 & - 4\,c^3f & - 6\,c^2ff & - 4\,cf^3 & - f^4 \end{array}\right\}$$

will alſo be $= 5000$.

Now

Now let all the terms in this equation that involve either ff, f^3, or f^4, be expunged from it. And we shall then have the compound quantity

$$\left\{\begin{array}{rrrr} & 14{,}937\,c & + & 14{,}937\,f \\ - & 1998\,cc & - & 3996\,cf \\ + & 80\,c^3 & + & 240\,c^2 f \\ - & c^4 & - & 4\,c^3 f \end{array}\right\} \text{nearly} = 5000.$$

But it has been shewn in the last article that $14{,}937\,c - 1998\,cc + 80\,c^3 - c^4$, or $14{,}937 \times 12.756{,}39 - 1998 \times \overline{12.756{,}39}|^2 + 80 \times \overline{12.756{,}39}|^3 - \overline{12.756{,}39}|^4$, is $= 5000.273{,}805{,}270{,}591{,}690{,}609{,}59$; and $3996\,c$ is $(= 3996 \times 12.756{,}39) = 50{,}974.534{,}44$; and

$240\,c^2$ is $(= 240 \times 162.725{,}485{,}832{,}1)$
$= 39{,}054.116{,}599{,}704{,}0$;

and $4\,c^3$ is $(= 4 \times 2075.789{,}760{,}213{,}742{,}119)$
$= 8303.159{,}040{,}854{,}968{,}476$.

Therefore the compound quantity

$$\left\{\begin{array}{l} 5000.273{,}805{,}270{,}591{,}690{,}609{,}59 \\ \qquad + 14{,}937\,f \\ \qquad - 50{,}974.534{,}44 \times f \\ \qquad + 39{,}054.116{,}599{,}704{,}0 \times f \\ \qquad - 8303.159{,}040{,}854{,}968{,}476 \times f \end{array}\right\}$$

will be, nearly, $= 5000$, or the compound quantity $5000.273{,}805{,}270{,}591{,}690{,}609{,}59 + 53{,}991.116{,}599{,}704{,}0 \times f - 59{,}277.693{,}480{,}854{,}968{,}476 \times f$ will be nearly $= 5000$, or the compound quantity $5000.273{,}$

805,

805,270,591,690,609,59 — 5,286.576,881,150,968,476 × f will be, nearly, = 5000.

Therefore (adding 5,286.576,881,150,968,476 × f to both sides,) we shall have 5000.273,805,270,591,690,609,59 = 5000 + 5,286.576,881,150,968,476 × f, and (subtracting 5000 from both sides,) 5,286.576,881,150,968,476 × f = 0.273,805,270,591,690,609,59; which is a simple equation properly prepared for resolution.

This equation is to be resolved by the single operation of dividing both sides of it by 5,286.576,881,150,968,476, the co-efficient of the unknown quantity f; by which we shall find f to be $\left(= \frac{0.273,805,270,591,690,609,59}{5286.576,881,150,968,476}\right) =$ 0.000,051,792. Therefore $e + f$, or 12.756,39 + f, will be (= 12.756,39 + 0.000,051,792) = 12.756,441,792; that is, the 3d near value of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, which is obtained by this second process of Mr. Raphson's method of approximation, will be 12.756,441,792.

Q. E. I.

Of this number 12.756,441,792, the first ten figures, 12.756,441,79, are exact; it's more accurate value being 12.756,441,794,483,744,02, as has been seen in art. 37. But this degree of exactness may be attained by carrying this approximation by Mr. Raphson's method one step further; which may be done as follows.

Art. 41.

Art. 41. Let the number 12.756,441,792, obtained by the foregoing proceſs, be ſubſtituted inſtead of x in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to diſcover whether the reſult of ſuch ſubſtitution will be greater, or leſs, than 5000, or the abſolute term of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and, from that circumſtance, to determine whether the true value of x in that equation will be greater, or leſs, than the ſaid number.

Now, if x is = 12.756,441,792, we ſhall have

xx $(= \overline{12.756,441,792}|^2) = 162.726,807,192,684,171,264$,

and x^3 $(= \overline{12.756,441,792}|^3) = 2075.815,043,951,482,558,968,975,065,088$,

and x^4 $(= \overline{12.756,441,792}|^4) = 26,480.013,779,125,008,935,590,937,951,694,483,357,696$.

Therefore $14,937x$ will be $(= 14,937 \times 12.756,441,792) = 190,542.971,047,104$,

and $1998x^2$ will be $(= 1998 \times 162.726,807,192,684,171,264) = 325,128.160,770,982,974,185,472$,

and $80x^3$ will be $(= 80 \times 2075.815,043,951,482,558,968,975,065,088) = 166,065.203,516,118,604,717,518,005,207,040$;

and conſequently $14,937x + 80x^3$ will be (= 190,542.971,047,104 + 166,065.203,516,118,604,717,518,005,207,040) = 356,608.174,563,222,604,717,518,005,207,040, and $1998x^2 + x^4$ will be (= 325,128.160,770,982,974,185,472 + 26,480.013,779,125,008,935,590,937,

937,951,694,483,357,696) = 351,608.174,550,107,983, 121,062,937,951,694,483,357,696; and $14,937x + 80x^3 - 1988x^2 - x^4$ will be (=

$$\left\{\begin{array}{r} 356,608.174,563,222,604,717,518,005,207,040, \\ -\ 351,608.174,550,107,983,121,062,937,951,694, \\ 483,357,696 \end{array}\right\}$$

= 5000.000,013,114,621,596,455,067,255,345,516,642, 304; that is, the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be = 5000.000,013,114,621,596, 455,067,255,345,516,642,304; which is a little greater than 5000, or the abſolute term of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. And therefore, for the reaſons given in art. 29, we may conclude that the true value of x in that equation will be greater than 12.756,441,792.

A third Proceſs of Mr. Raphſon's Method of Approximation.

Art. 42. Having now determined that the number 12.756,441,792, which was the near value of x obtained by the ſecond proceſs of Mr. Raphſon's method of approximation, is leſs than it's true value, let the ſaid near value 12.756,441,792 be denoted by the letter d, and the ſmall quantity by which the true value of x exceeds it be denoted by the letter g.

Then we ſhall have $x = d + g$, and conſequently xx ($= \overline{d+g}|^2$) $= dd + 2dg + gg$, and x^3 ($= \overline{d+g}|^3$) $= d^3 + 3d^2g + 3dg^2 + g^3$, and x^4 ($= \overline{d+g}|^4$) $= d^4 + 4d^3g + 6d^2g^2 + 4dg^3 + g^4$. Therefore

Therefore $14{,}937\,x$ will be $(= 14{,}937 \times \overline{d+g})$ $= 14{,}937\,d + 14{,}937 \times g$; and $1998\,x^2$ will be $(= 1998 \times \overline{dd + 2dg + gg}) = 1998\,dd + 3996\,dg + 1998\,gg$; and $80\,x^3$ will be $(= 80 \times \overline{d^3 + 3d^2g + 3dg^2 + g^3}) = 80\,d^3 + 240\,d^2g + 240\,dg^2 + 80\,g^3$. And confequently the whole compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$ will be equal to the compound quantity

$$\left\{\begin{array}{l} 14{,}937\,d + 14{,}937\,g \\ -\ 1998d^2 - 3996\,dg - 1998g^2 \\ +\ 80\,d^3 + 240\,d^2g + 240\,dg^2 + 80g^3 \\ -\ d^4 - 4\,d^3g - 6\,d^2g^2 - 4\,dg^3 - g^4. \end{array}\right\}$$

But the former compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$ is equal to 5000.

Therefore the latter compound quantity

$$\left\{\begin{array}{l} 14{,}937\,d + 14{,}937\,g \\ -\ 1998d^2 - 3996\,dg - 1998g^2 \\ +\ 80\,d^3 + 240\,d^2g + 240\,dg^2 + 80g^3 \\ -\ d^4 - 4\,d^3g - 6\,d^2g^2 - 4dg^3 - g^4 \end{array}\right\}$$

will alfo be equal to 5000.

Now let the fix terms $-\ 1998\,g^2 + 240\,dg^2 - 6d^2g^2 + 80\,g^3 - 4\,dg^3 - g^4$ (which involve in them the fquare, cube, and fourth power of the unknown quantity g,) be expunged out of this equation, on account of their fmallnefs in comparifon of the four terms

+

$+ 14{,}937\,g - 3996\,dg + 240\,d^2g - 4\,d^3g$, which involve the simple power of g. And we shall then have the remaining compound quantity

$$\left\{\begin{array}{rcr} 14{,}937\,d & + & 14{,}937\,g \\ -\ 1998\,d^2 & - & 3996\,dg \\ +\ \ 80\,d^3 & + & 240\,d^2g \\ -\ \ d^4 & - & 4\,d^3g \end{array}\right\} \text{ nearly } = 5000.$$

But we have seen, in the last article, that, if d is equal to 12.756,441,792, the quadrinomial quantity $14{,}937\,d - 1998\,d^2 + 80\,d^3 - d^4$ will be equal to 5000.000, 013,114,621,596,455,067,255,345,516,642,304.

Therefore the compound quantity 5000.000,013,114, 621,596,455,067,255,345,516,642,304 $+ 14{,}937\,g - 3996\,dg + 240\,d^2g - 4\,d^3g$ will be nearly $= 5000$.

But, since d is $=$ 12.756,441,792, we shall have d^2 ($= \overline{12.756{,}441{,}792}|^2$) $=$ 162.726,807,192,684,171, 264, and d^3 ($= \overline{12.756{,}441{,}792}|^3$) $=$ 2075.815,043, 951,482,558,968,975,065,088, and $3996\,d$ ($= 3996 \times$ 12.756,441,792) $=$ 50,974.741,400,832, and $240\,d^2$ ($= 240 \times$ 162.726,807,192,684,171,264) $=$ 39,054. 433,726,244,201,103,360, and $4\,d^3$ ($= 4 \times$ 2075.815, 043,951,482,558,968,975,065,088) $=$ 8303.260,175, 805,930,235,875,900,260,352.

Therefore the compound quantity 5000.000,013,114, 621,596,455,067,255,345,516,642,304 $+ 14{,}937\,g - 3996\,dg + 240\,d^2g - 4\,d^3g$ will be $=$ 5000.000, 013,

013,114,621,596,455,067,255,345,516,642,304 + 14,937 g — 50,974·741,400,832 × g + 39,054·433,726,244,201,103,360 × g — 8303·260,175,805,930,235,875,900,260,352 × g = 5000.000,013,114,621,596,455,067,255,345,516,642,304

+ 53,991·433,726,244,201,103,360 × g

— 59,278.001,576,637,930,235,875,900,260,352 × g

= 5000,000,013,114,621,596,455,067,255,345,516,642,304

— 5286.567,850,393,729,132,515,900,260,352 × g.

Therefore this last quantity 5000.000,013,114,621,596,455,067,255,345,516,642,304 — 5286.567,850,393,729,132,515,900,260,352 × g will be nearly = 5000; and consequently (adding 5286.567,850,393,729,132,515,900,260,352 × g to both sides,) 5000.000,013,114,621,596,455,067,255,345,516,642,304 will be = 5000 + 5286.567,850,393,729,132,515,900,260,352 × g, and, lastly, (subtracting 5000 from both sides,) 5,286.567,850,393,729,132,515,900,260,352 × g will be = 0.000,013,114,621,596,455,067,255,345,516,642,304; which is a simple equation properly prepared for resolution.

Therefore g will be =

$$\frac{0.000,013,114,621,596,455,067,255,345,516,642,304}{5286,567,850,393,729,132,515,900,260,352} =$$

0.000,000,002,480,744,022,88; and consequently $d + g$, or 12.756,441,792 + g, will be (= 12.756,441,792 + 0.000,000,002,480,744,022,88) = 12.756,441,794,480,

744,022,88; that is, the fourth near value of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, obtained by this third process of Mr. Raphson's method of approximation, will be $= 12.756,441,794,480,744,022,88$. Q. E. I.

Art. 43. This number, 12.756,441,794,480,744,022,88, agrees with the number found by Dr. Wallis and Dr. Halley, for the value of x, to wit, the number 12.756, 441,794,480,744,02, in all it's nineteen figures; and it agrees with the number found above in art. 37 by Dr. Halley's method of approximation, to wit, the number 12.756,441,794,480,744,022,60, in the first twenty figures 12.756,441,794,480,744,022: and therefore those twenty figures are probably exact, and the true value of x is greater than 12.756,441,794,480,744,022, but less than 12.756,441,794,480,744,023. But this cannot be proved with certainty without substituting those numbers instead of x in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$; which would be a work of great labour, and is not necessary to the object of this discourse, which is to make a comparison between Dr. Halley's and Mr. Raphson's methods of resolving high Algebräick equations by approximation, and to enable the reader to form a judgement concerning their respective merits, and determine for himself to which of the two he will give the preference: and this object, I hope, is now attained.

A SCHO-

A SCHOLIUM,

Concerning another Method of approximating to the Roots of High Equations, that may be called The Differential Method.

Art. 44. Having now gone through the resolution of the biquadratick equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$ by both Dr. Halley's and Mr. Raphson's methods of approximation, and obtained the value of one of it's roots to a great degree of exactness, I will here state to the reader another method that might have been taken to investigate the same root to the same degree of exactness, that is totally different from both the foregoing methods. This method (which, I think, may be properly distinguished by the appellation of the *Differential* method of approximating to the roots of high equations,) occurred to me as a natural and convenient artifice for obtaining the number 12.7, consisting of three figures, as the value of a, or the first near value of x in the aforesaid equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$, which we were afterwards to make use of as the basis, or ground-work, of the further approaches which were intended to be made to the true value of x in that equation by Dr. Halley's and Mr. Raphson's methods of approximation. For the steps we took in art. 27 to find the said number a, or 12.7, for a first near value of x, and the basis of the subsequent approximations, were as follows:

Art. 45. In the firſt place we made a random gueſs that the number 10 (which is a number very eaſily managed in calculation,) might not be very different from the value of x in the propoſed equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$; and then, in order to try the juſtneſs of this conjecture, we ſubſtituted 10 inſtead of x in the compound quantity $14,937x - 1998xx + 80x^3 - x^4$; and we found that the value of the ſaid quantity reſulting from this ſubſtitution was 19,570; which is conſiderably greater than 5000, or the abſolute term of the propoſed equation. We therefore concluded that the true value of x in that equation muſt be greater, or leſs, than 10 by more than a ſingle unit, and we conjectured, in the ſecond place, that it might be nearly equal to 12.

We then tried this ſecond conjecture by ſubſtituting 12 inſtead of x in the compound quantity $14,937x - 1998xx + 80x^3 - x^4$, and we found the value of the ſaid quantity reſulting from this ſubſtitution to be 9,036; which is leſs than half the former reſult 19,570, but yet is conſiderably greater than the abſolute term 5000.

We therefore made a third conjecture that x might be nearly equal to 13, and we ſubſtituted 13 inſtead of x in the compound quantity $14,937x - 1998xx + 80x^3 - x^4$. And the value of the ſaid quantity reſulting from this ſubſtitution was found to be 3,718; which is leſs than the abſolute term 5000.

We then obſerved that, ſince, when x was equal to 12, the compound quantity $14,937x - 1998xx +$

$80x^3 - x^4$ was equal to 9,036, and, when x was equal to 13, the ſame compound quantity was equal to 3,718, it followed that, while x increaſed from 12 to 13, the ſaid compound quantity muſt have decreaſed from 9,036 to 3,718, and conſequently muſt, at ſome point of time during the ſaid decreaſe, have been equal to the intermediate magnitude 5000, or the abſolute term of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. And therefore we concluded that there muſt be ſome value of x greater than 12, but leſs than 13, which would make the ſaid compound quantity $14,937x - 1998xx + 80x^3 - x^4$ be exactly equal to the intermediate magnitude 5000; or, in other words, that the true value of x in the equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$ muſt be greater than 12, but leſs than 13.

Art. 46. Having thus found that x muſt be greater than 12, but leſs than 13, we might have taken either of thoſe numbers for the baſis of a further approximation to the true value of x by either of the two methods of Dr. Halley and Mr. Raphſon. But, as I was deſirous of approaching a little nearer to the true value of x before I began the application of either of thoſe methods, I made uſe of the following conjectural and probable ſuppoſition, (which is ſimilar to that by which the logarithm of a number that is of an intermediate magnitude between two numbers that are very nearly equal to each other, and of which the logarithms are known, is derived from the logarithms of the ſaid two extreme numbers,) to wit, that, ſince the number 12, the number x, (or the

true value of the root of the propofed equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$,) and the number 13 were three numbers that did not much differ from each other, and the refults of the fubftitutions of thofe three numbers inftead of x in the compound quantity $14,937x - 1998xx + 80x^3 - x^4$ were 9036, 5000, and 3718, the difference of the firft number 12 and the third number 13 would be to the difference of the firft number 12 and the fecond number x in nearly the fame proportion as the difference of the firft refult 9036, (correfponding to the firft number 12) and the third refult 3,718, (correfponding to the third number 13) to the difference of the firft refult 9036, (correfponding to the firft number 12,) and the fecond refult 5000, (correfponding to the fecond number x,) that is, that $13 - 12$ would be to $x - 12$ in nearly the fame proportion as $9,036 - 3,718$ to $9036 - 5000$, or that 1 would be $x - 12$ in nearly the fame proportion as 5318 to 4036; whence it followed that $x - 12$ would be nearly $(= \frac{1 \times 4036}{5318} = \frac{4036}{5318}) = 0.7$, and therefore that x would be nearly $(= 0.7 + 12) = 12.7$; which are the three firft figures of the true value of x, (the faid true value having been found to be 12.756,441,794,480,744,022,) and therefore feemed to be fufficiently near the faid true value to be taken for the bafis of the further approximations intended to be made afterwards to the faid true value by the methods of Dr. Halley and Mr. Raphfon.

Art. 47. But I afterwards found, on examining this proportion more attentively, and continuing the divifion of

of 4036 by 5318 to three figures in the quotient, that, if I had wifhed to obtain the value of x to a greater degree of exactnefs, before I had begun the further approximations to it's true value by Dr. Halley's and Mr. Raphfon's methods, the fame proportion would have given me 0.758 for the value of $x - 12$, and confequently 12.758 for the firft near value of x; of which number the four firft figures 12.75 are exact, or agree with the four firft figures of the true value of x, to wit, 12.756,441,794,480,744,022, and the fifth figure 8 is greater than the truth by only 2, or the 4th part of the faid fifth figure, or the 6378th part of the faid true value of x. This would have been a very great degree of exactnefs to have been attained at once by this proportion which I had conjectured to have fubfifted in fome degree between the differences of the faid three numbers 12, x, and 13, and the differences of the correfponding refults of the fubftitution of the faid numbers in the compound quantity $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4$: and therefore it appears that this conjecture is a very happy one, and approaches very nearly to the truth.

Art. 48. This degree of exactnefs in the near value of x, obtained by means of this proportion, was, I confefs, much greater than I had expected, the faid conjecture having been made with the hope of finding only one figure in the faid quotient to have been exact. But when I had obferved that the two firft figures of the faid quotient, to wit, the figures 0.75, were exact, and that the third figure 8 was not very much greater than the correfponding figure 6 of the true value of $x - 12$, I began to

think

think that, if x and the two numbers that were nearly equal to it were to be taken much nearer to an equality with each other than the three numbers 12, x, and 13, (of which the greateſt exceeded the leaſt by 1, or the 12th part of the leaſt,) the aforeſaid ſuppoſed proportionality of the differences of the three contiguous numbers to the differences of the reſults of their ſubſtitution in the compound quantity $14,937\,x - 1998\,xx + 80\,x^3 - x^4$ would approach ſtill nearer to the truth, and conſequently that the number of figures that would be exact in the value of x obtained by means of ſuch ſuppoſed proportionality would be greater than it was in the former caſe. I therefore reſolved to try the effect of a ſecond conjecture founded on this ſuppoſed proportionality between the differences of x and two numbers very nearly equal to it and the differences of the three corresponding reſults of the ſubſtitution of x and the ſaid contiguous numbers in the ſaid compound quantity $14,937\,x - 1998\,xx + 80\,x^3 - x^4$. And with this view I took the near value of x obtained by the former proportion, to wit, the number 12.758, for the greater of the two contiguous numbers to x, and the number 12.756 for the leſſer of the ſaid two contiguous numbers, and I ſubſtituted, 1ſt, the number 12.758, and, 2ndly, the number 12.756, inſtead of x in the ſaid compound quantity $14,937\,x - 1998\,xx + 80\,x^3 - x^4$. The value of this compound quantity reſulting from the ſubſtitution of 12.758 was 4,991.762,652,593,904; and the value of it reſulting from the ſubſtitution of 12.756 was 5002.335,593,512,704. Therefore, while x increaſes from 12.756 to 12.758, the compound quantity $14,937\,x$

—

— 1998 xx + 80 x^3 — x^4 will decreafe from 5002.335, 593,512,704 to 4991.762,652,593,904, and confequently will, at fome one inftant of time during the faid decreafe, be equal to the intermediate magnitude 5000; or, in other words, the true value of x in the propofed equation 14,937 x — 1998 x^2 + 80 x^3 — x^4 = 5000 will be greater than 12.756, but lefs than 12.758.

Here therefore we have three contiguous numbers, to wit, 12.756, x, and 12.758, the greateft of which exceeds the leaft by only 0.002, or the 6378th part of the leaft number 12.756; and we have the three correfponding quantities 5002.335,593,512,704, 5000, and 4991.762,652,593,904, which refult from the fubftitution of the faid three contiguous numbers in the faid compound quantity 14,937 x — 1998 xx + 80 x^3 — x^4. We may therefore fuppofe that the difference of the firft and third of the faid three numbers 12.756, x, and 12.758, will be to the difference of the firft and fecond of the faid three numbers, to wit, 12.756 and x, in nearly the fame proportion as the difference of the firft and third of the faid refults (correfponding to the faid three numbers,) is to the difference of the firft and fecond of the faid refults, or that 12.758 — 12.756 will be to x — 12.756 in nearly the fame proportion as 5002.335,593,512,704 — 4991.762,652,593,904 is to 5002.335,593,512,704 — 5000, or that 0.002 will be to x — 12.756 in nearly the fame proportion as 10.572,940,918,800 is to 2.335,593,512,704; whence it will follow that x — 12.756 will be nearly

$$\left(= \frac{0.002 \times 2.335,593,512,704}{10.572,940,918,800} = \frac{0.004,671,187,025,408}{10.572,940,918,800}\right)$$

=

$= 0.000,441,805$, and therefore that x will be $(= 0.000,441,805 + 12.756) = 12.756,441,805$; of which value of x the first eight figures 12.756,441 are exact, and the ninth figure 8 is greater than the corresponding figure 7 in the true value of x, (which is 12.756,441, 794,480,744,022) by only an unit in the said ninth figure, or 0.000,000,1.

Art. 49. The substitutions of 12.758 and 12.756 instead of x in the compound quantity $14,937x - 1998xx + 80x^3 - x^4$ will be as follows:

If x is $= 12.758$, we shall have xx $(= \overline{12.758}|^2)$ $= 162.766,564$, and x^3 $(= \overline{12.758}|^3) = 2076.575,823,512$, and x^4 $(= \overline{12.758}|^4) = 26,492.954,356,366,096$, and consequently $14,937x$ $(= 14,937 \times 12.758) = 190,566.246$, and $1998xx$ $(= 1998 \times 162.766,564) = 325,207.594,872$, and $80x^3$ $(= 80 \times 2076.575,823,512) = 166,026.065,880,960$. Therefore the whole compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will, upon this supposition, be

$$\begin{array}{llll} (= & 190,566.246 & - & 325,207.594,872 \\ + & 166,126.065,880,960 & - & 26,492.954,356,366,096 \\ = & 356,692.311,880,960 & - & 351,700.549,228,366,096) \end{array}$$

$= 4991.762,652,593,904$; which is somewhat less than 5000, or the absolute term of the equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$.

And,

And, if x is $= 12.756$, we shall have xx ($= \overline{12.756}|^2$) $= 162.715,536$, and x^3 ($= \overline{12.756}|^3$) $=$ $2075.599,377,216$, and x^4 ($= \overline{12.756}|^4$) $= 26,476.345,655,767,296$, and consequently $14,937x$ ($= 14,937 \times 12.756$) $= 190,536.372$, and $1998x^2$ ($= 1998 \times 162.715,536$) $= 325,105.640,928$, and $80x^3$ ($= 80 \times 2075.599,377,216$) $= 166,047.950,177,280$. Therefore the whole compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will, upon this supposition, be

$$\begin{array}{lll} (= 190,536.372 & - & 325,105.640,928 \\ \;\; + 166,047.950,177,280 & - & 26,476.345,655,767,296 \\ \;\; = 356,584.322,177,280 & - & 351,581.986,583,767,296) \end{array}$$

$= 5002.335,593,512,704$; which is a little greater than 5000, or the absolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Art. 50. Since the former near value of x, to wit, 12.758, (which was obtained by means of the substitutions of the small numbers 12 and 13 instead of x in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, and of the proportion derived from those substitutions,) was exact in the four first figures 12.75, and the last near value of x, to wit, 12.756,441,805, (which has been obtained by means of the substitutions of the numbers 12.758 and 12.756 instead of x in the said compound quantity, and of the proportion derived from those substitutions,) is exact in the eight first figures 12.756,441, and but a little too great in the ninth figure 8, we may conclude that every new process of this differential method of resolving equations will double the number

number of figures in the value of x, or the root sought, that are already known; and therefore, in point of exactness, this method will be nearly upon a level with Mr. Raphson's method of approximation, by which the figures of the root that are already known are doubled by every new process. Nor is there much difference in the labour of performing the necessary calculations in this differential method of approximating to the roots of equations and in that of Mr. Raphson, which consists in substituting the binomial quantity $a + e$, or $a - e$, instead of x in the proposed equation, and then resolving the transformed equation, arising from such substitution, as if it were a mere simple equation, or neglecting all the terms of it that involve in them any other power of the new unknown quantity e than it's simple power, or e itself. For in both methods it is necessary to substitute the former near value of x, to wit, the number 12.758, instead of x in the terms of the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$; and in this *differential* method we are also obliged to substitute another near value of x, differing from the former near value 12.758 in the fifth, or last, figure, to wit, the number 12.756, instead of x in the terms of the said compound quantity; the labour of making which second substitution seems to be about equal to, or, perhaps, a little greater than, that of computing the compound co-efficient of e in the transformed equation

$$\left\{\begin{array}{rrrl} & 14,937a & - & 14,937e \\ - & 1998a^2 & + & 3996ae - \&c \\ + & 80a^3 & - & 240a^2e + \&c \\ - & a^4 & + & 4a^3e - \&c \end{array}\right\} = 5000,$$

which

which would have arisen, in Mr. Raphson's method of proceeding, from the substitution of $12.758 - e$, or $a - e$, instead of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, to wit, the co-efficient $-14,937 + 3996a - 240a^2 + 4a^3$, or $-14,937 + 3996 \times 12.758 - 240 \times \overline{12.758}|^2 + 4 \times \overline{12.758}|^3$, or $-14,937 + 3996 \times 12.758 - 240 \times 162.766,564 + 4 \times 2076.575,823,512$. But Mr. Raphson's method seems to be more perspicuous and scientifick than the other, as well as, in some degree, less laborious: and therefore, upon the whole, I think it preferable to this differential method of approximation, as well as to that of Dr. Halley, excepting in the beginning of the resolution of an equation, or when we are endeavouring to find the quantity a, or the first near value of x, which is afterwards to be made the basis, or ground-work, of a further approach to the true value of x by either Dr. Halley's or Mr. Raphson's method of approximation: for to this purpose I think the said differential method of approximation seems to be remarkably well adapted.

Art. 51. And in this use of the said differential method for the discovery of a, or the first near value of x, which is to be made the basis of a further approach to it's true value by Dr. Halley's or Mr. Raphson's methods of approximation, I should think it would be adviseable to use as many figures of the value of x, obtained by it, as we have reason to think are exact, or nearly exact. Thus, for example, in resolving the foregoing equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$, I think it would have

have been better to make a equal to the number 12.758, consisting of five figures, obtained by the first easy proportion of 1 to $x - 12$ as 5318 to 4036, than to make it equal to the number 12.7, consisting of only three figures, which was obtained from the same proportion by taking only the first figure 0.7 of the quotient of the division of 4036 by 5318. For, if we had taken a equal to all the said five figures 12.758, we might, by the application of only one process of Mr. Raphson's method of approximation, have obtained a second near value of x that would have been exact to at least eight figures. And by this union of this differential method of approximating to the roots of high equations with Mr. Raphson's method of resolving them, by employing the former method in the first stage of the resolution to obtain to as great a degree of exactness as one process of the said differential method will enable us to find it, the value of a, or the first near value of the root sought, and then proceeding to make use of a, or the near value of the root x so obtained, as a basis for a further approximation to the true value of the said root by Mr. Raphson's method, the calculator will, as I conceive, obtain the value of the root sought to any proposed degree of exactness with the least trouble and the least perplexity possible.

Art. 52. I will observe, however, that the learned Dr. Hutton, of Woolwich Academy (in his late compendious collection of tracts on the different branches of the Mathematicks in two volumes, octavo, drawn-up for the use of the Cadets in the Military Academy at Woolwich,) recommends this differential method of resolving high

equations

equations by approximation (which he calls *The Method of Trial and Error*, or *The Double Rule of False Position*,) for the compleat resolution of such equations, independently of, and in preference to, all other methods of resolving them whatsoever.

End of the Scholium begun in Art. 44, *page* 97.

Art. 53. The foregoing biquadratick equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$ has three other roots, or, in the language of modern Algebräists, three other real and affirmative roots, of which the least is less than 1, and is equal to 0.350,987,04, &c, and the other two are greater than 30, being equal to 32.060,290, &c, and 34.832,280, &c, as I have found upon an investigation of them by Mr. Raphson's method of approximation, if no mistakes have been made in the calculations that were necessary for that purpose. The investigation of the least of these three roots, to wit, 0.350,987,04 &c, is given in the 3d volume of the *Scriptores Logarithmici*, pages 722, 723, 724.

Mr. Raphson's Observations on Monsieur de Lagny's and Dr. Halley's claims to the merit of having invented their Methods of resolving Equations by Approximation.

Art. 54. Before I conclude this discourse I will insert two passages from Mr. Raphson's appendix to the second edition of his ingenious and useful treatise on the resolution of equations of all orders by approximation, intitled *Analysis Æquationum Universalis*, in which he puts-in his claim to the merit of having invented this method of resolving

resolving equations, and declares that he had had the same thought which Monsieur de Lagny and Dr. Halley afterwards pursued, of making use of the terms that involve the square and other higher powers of the unknown quantity contained in the transformed equation, in order to obtain the value of the said unknown quantity, and consequently the value of x, or the root of the original equation, to a greater degree of exactness; but that, upon mature consideration, he had laid it aside, as tending to introduce new difficulties into the investigation, and to diminish it's perspicuity and simplicity. His words are as follows: *An Dominus* de Lagny *librum meum unquam viderat, nec-ne, prorsùs nescio. Quibus-ce modis non solùm sua* [*ejus*] *methodus, sed et etiàm aliæ quam plurimæ, eodem prorsùs processu, et perpetuâ inde derivatâ graduum scalâ, inveniri possint, hujus-ce appendicis est ostendere; idque quàm possimus brevissimè.*

Ipse equidem de gradatìm inferendis, quas priùs rejeceram in Theoremate Vietæo, *potestatibus, olìm cogitavi: sed tamen non prosecutus fui; utpote qui methodum meam, harum omnium fundamentalem, veluti facillimam semper existimavi. Subsequenti processu earum omnium inventionem indagare cuilibet liceat.* See Mr. Raphson's *Analysis Æquationum Universalis*, edition 2nd, A. D. 1697, page 49. And again, in page 55, he concludes his appendix with these words: *Innumeras etiàm alias methodos et abbreviationes,* novarum *quidem* methodorum *nomine insigniendas, adinvenire liceat; quæ tamen omnia fundamentali huic superiorum potestatum, imprimis, rejectionis methodo, posteàque gradatìm retinendarum, innitantur. Nostram tamen simplicissimam fore et facillimam, cuivis pateat.*

OF

OF THE EXTRACTION OF THE SQUARE-ROOT OF A GIVEN NUMBER BY APPROXIMATION.

Art. 55. I HAVE now accomplifhed the object I had principally in view in drawing-up this Appendix to the foregoing Tract of Dr. Halley; which was to make a comparifon between his method of refolving cubick and other higher affected equations by approximation, which proceeds by the refolution of quadratick equations, and Mr. Raphfon's method of refolving them, which proceeds by the refolution of only fimple equations. And my readers will now, I hope, be able to determine for themfelves " to which of the two methods they ought to give the preference:" or, if the preference ought in fome cafes to be given to Dr. Halley's method, and in others to Mr. Raphfon's, to determine " in what cafes Dr. Halley's method, and in what cafes Mr. Raphfon's method, deferves to be preferred." For my own part, I am decidedly of opinion (as I have declared above in page 27,) that in moft cafes Mr. Raphfon's method of approximation will be found more convenient than Dr. Halley's, though there may, perhaps, be fome few occafions on which it may be expedient to have recourfe to Dr. Halley's method. Now, whenever it fhall be judged convenient to proceed by Dr. Halley's method, it will evidently be neceffary, in every feparate procefs of the approximation, to extract the fquare-root of a given number;

number; and in the fecond and other following proceffes of fuch approximations by Dr. Halley's method, it is obvious that the given numbers of which the fquare-roots muft be extracted, will be very long numbers, confifting often of 12 or 14, or 16 or more, decimal figures. In thefe cafes therefore it may feem proper to inquire, whether the labour of extracting thefe fquare-roots may not be leffened by applying the method of approximation even to that well-known, and not very difficult, operation, inftead of performing it in the common way: becaufe in extracting the fquare-root of a number in the common way we obtain only one new figure of the fquare-root fought at every new divifion of the laft remainder by the new divifor formed by doubling the root already found; whereas it may be fuppofed that, by fome of the methods of approximation, we might obtain feveral new figures of the root fought at every new operation, and thereby avoid much unneceffary labour. Now, in anfwer to this inquiry, I believe, it may fafely be affirmed, that, in general, it will be found moft convenient to extract the fquare-root of a given number in the common way, without having recourfe to any of the methods of approximation; and more efpecially, when it is propofed to extract it to only fix, or feven, or even eight, figures: and further, that, if ever it fhould be judged expedient, when the fquare-root has already been found, either in the common way, or by means of a table of logarithms, exact to feven, or eight, places of figures, to carry the extraction of it to feven or eight figures more, or to fifteen or fixteen figures in all, it will be much better to make ufe of the expreffion $a + \frac{b}{2a}$, which is afforded

us

us for this purpose by Mr. Raphson's method of approximation, and is derived from the resolution of a simple equation, than to have recourse to the expression $a + \frac{ab}{2aa + \frac{b}{2}}$, which is given us above by Dr. Halley in the foregoing tract, page 9, and is derived from the contemplation and imperfect resolution of a quadratick equation. These things will become evident from the investigation of the two expressions $a + \frac{b}{2a}$ and $a + \frac{ab}{2aa + \frac{b}{2}}$, and the application of them to the extraction of the square-root of a particular number; in order to which it will be necessary to give a solution of the following Problem.

A PROBLEM.

Art. 56. To extract the square-root of a given number by approximation.

SOLUTION.

Let the given number, of which the square-root is to be extracted, and which we will suppose to be greater than 1, be called N; and let it's unknown square-root be called x.

Then

Then will xx be $= N$.

Let aa be any ſquare number leſs than N, of which the ſquare-root a is known; and let the exceſs of N above aa be called b.

Then will N be $= aa + b$; and conſequently xx, which is $= N$, will alſo be $= aa + b$; and, therefore, x will be $= \sqrt{aa + b}$.

Now, ſince $aa + b$ is greater than aa, it follows that the ſquare-root of $aa + b$ will be greater than the ſquare-root of aa; that is, $\sqrt{aa + b}$ will be greater than a. Therefore x will be greater than a.

Further, ſince x is greater than a, let it's exceſs above a be called z.

Then will x be $= a + z$, and conſequently xx will be $(= \overline{a + z}|^2) = aa + 2az + zz$.

But $xx = aa + b$.

Therefore $aa + 2az + zz$ will alſo be $= aa + b$; and conſequently (ſubtracting aa from both ſides,) $2az + zz$ will be $= b$.

But z is much leſs than a, and, *à fortiori*, than $2a$. Therefore zz will be much leſs than $2az$, and conſequently $2az$ alone will be nearly equal to $2az + zz$, or to b. And conſequently z will be nearly $= \frac{b}{2a}$,

and

and $a + z$ will be nearly $= a + \frac{b}{2a}$; or $a + \frac{b}{2a}$ will be a near value of $a + z$, or x, or $\sqrt{aa + b}$, or $\sqrt{N}$. Q. E. I.

This expreſſion $a + \frac{b}{2a}$ is Mr. Raphſon's approximation to the value of $\sqrt{aa + b}$, or $\sqrt{N}$, or the ſquare-root of the given number N. It is evidently ſomewhat greater than the truth; becauſe $2az + zz$ is $= b$, and conſequently $2az$ is accurately $= b - zz$, and z is accurately $= \frac{b}{2a} - \frac{zz}{2a}$, which is leſs than $\frac{b}{2a}$. But it will uſually give us as many new figures of $a + z$, or the value of $\sqrt{aa + b}$, or $\sqrt{N}$, or, at leaſt, as many new figures of the ſaid root, wanting one, exactly, as there are figures in a, the part of the root that is already known.

The Inveſtigation of Dr. Halley's Expreſſion for the ſame Purpoſe.

Art. 57. The approximation to the value of $\sqrt{aa + b}$, or $\sqrt{N}$, given us by Monſieur *de Lagny* and Dr. Halley, is $a + \cfrac{b}{2a + \cfrac{b}{2a}}$, or $a + \cfrac{ab}{2aa + \cfrac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, and may be found as follows:

Since

Since $2az + zz$ is $= b$, and $2az + zz$ is $= z \times \overline{2a + z}$, we shall have $z \times \overline{2a + z} = b$, and consequently $z = \frac{b}{2a + z}$.

But z is nearly equal to $\frac{b}{2a}$, as we have seen in the last article.

Therefore $\frac{b}{2a + z}$ will be nearly equal to $\frac{b}{2a + \frac{b}{2a}}$, or to $\frac{ab}{2aa + \frac{b}{2}}$, or to $\frac{2ab}{4aa + b}$. Therefore z (which is equal to $\frac{b}{2a + z}$,) will be nearly equal to $\frac{b}{2a + \frac{b}{2a}}$, or to $\frac{ab}{2aa + \frac{b}{2}}$, or to $\frac{2ab}{4aa + b}$; and consequently $a + z$, or $\sqrt{aa + b}$, or $\sqrt{}$N, will be nearly equal to $a + \frac{b}{2a + \frac{b}{2a}}$, or to $a + \frac{ab}{2aa + \frac{b}{2}}$ (which is Dr. Halley's expression,) or to $a + \frac{2ab}{4aa + b}$.

Q. E. I.

Art. 58. This near value of $\sqrt{aa + b}$ is always less than the truth; because $\frac{b}{2a}$ (which forms the second

term

term of the binomial denominator of the fraction $\frac{b}{2a + \frac{b}{2a}}$,) is greater than z, (or the second term of the binomial denominator of the fraction $\frac{b}{2a + z}$,) and consequently the fraction $\frac{b}{2a + \frac{b}{2a}}$ must be less than the fraction $\frac{b}{2a + z}$, and $a + \frac{b}{2a + \frac{b}{2a}}$ must be less than $a + \frac{b}{2a + z}$, or than $a + z$, or $\sqrt{aa + b}$, or $\sqrt{N}$. Q. E. D.

Art. 59. This expression $a + \frac{b}{2a + \frac{b}{2a}}$, or $a + \frac{ab}{2aa + \frac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, will approach much nearer than the former expression $a + \frac{b}{2a}$ to the true value of $\sqrt{aa + b}$. But it will be much more difficult to compute than the expression $a + \frac{b}{2a}$, on account of the much greater number of figures in the denominator, or divisor, $2a + \frac{b}{2a}$, or $2aa + \frac{b}{2}$, or $4aa + b$, than in the denominator, or divisor, $2a$. And this difficulty is so great when a is a number consisting of seven, or

eight,

eight, figures, that I consider this expression $a + \frac{b}{2a + \frac{b}{2a}}$, or $a + \frac{ab}{2aa + \frac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, as of very little use in these cases.

A Third Expression for the same Purpose.

Art. 60. But the solution of the foregoing problem will furnish us with another expression for the value of x, or $\sqrt{aa + b}$, that will approach almost as nearly to it's true value as the foregoing expression $a + \frac{b}{2a + \frac{b}{2a}}$, or $a + \frac{ab}{2aa + \frac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, given us by Monsieur de Lagny and Dr. Halley, and will be somewhat less difficult to compute than that expression of Dr. Halley, though much more so than Mr. Raphson's expression $a + \frac{b}{2a}$, and indeed too much so to make it worth our while on most occasions to have recourse to it. This expression is $a + \frac{b}{2a} - \frac{\frac{bb}{4aa}}{2a}$, or $a + \frac{b}{2a} - \frac{bb}{2a \times 4aa}$, or $a + \frac{b}{2a} - \frac{bb}{8a^3}$, and may be found as follows:

Since $2az + zz$ is $= b$, we shall have $2az = b - zz$, and consequently $z = \frac{b}{2a} - \frac{zz}{2a}$. Therefore,

fore, if we ſubſtitute $\frac{b}{2a}$ (which is nearly equal to, but ſomewhat greater than z) inſtead of z in the term $\frac{zz}{2a}$, we ſhall have z nearly equal to, but ſomewhat greater than, $\frac{b}{2a} - \frac{\frac{bb}{4aa}}{2a}$, or $\frac{b}{2a} - \frac{bb}{2a \times 4aa}$, or $\frac{b}{2a} - \frac{bb}{8a^3}$. Therefore $a + z$ will be nearly equal to, but ſomewhat greater than, $a + \frac{b}{2a} - \frac{\frac{bb}{4aa}}{2a}$, or $a + \frac{b}{2a} - \frac{bb}{2a \times 4aa}$, or $a + \frac{b}{2a} - \frac{bb}{8a^3}$; that is, x, or $\sqrt{aa+b}$, will be very nearly equal to, but ſomewhat greater than, the trinomial quantity $a + \frac{b}{2a} - \frac{\frac{bb}{4aa}}{2a}$, or $a + \frac{b}{2a} - \frac{bb}{2a \times 4aa}$, or $a + \frac{b}{2a} - \frac{bb}{8a^3}$. Q. E. I.

Note. The three terms of this laſt expreſſion $a + \frac{b}{2a} - \frac{bb}{8a^3}$ are the ſame with the three firſt terms of the infinite ſeries for expreſſing the value of $\sqrt{aa+b}$, or $\overline{aa+b}|^{\frac{1}{2}}$, derived from Sir Iſaac Newton's binomial theorem.

Art. 61. We will now apply theſe different expreſſions to the inveſtigation of the ſquare-root of the number 32, or 25 + 7, (in which 25 anſwers to aa, and 7 to b, in the foregoing general expreſſion $aa + b$,) in order to try how far they will contribute to facilitate the extraction of that ſquare-root. And that the advantage ariſing from the uſe of them above the common method of extracting the ſquare-root (if there is any ſuch advantage,) may appear the more clearly, I will firſt exhibit the operation of extracting this ſquare-root to thirteen places of figures by the common method. Now this extraction will be as follows:

The

The Extraction of the Square-Root of the Number 32 by the Common Method.

32 (5.656,854,249,492,

25

106) ·7.00

636

1125) ·6400

5625

11306) ·77500

67836

113128) ·966400

905024

1131365) ·6137600

5656825

11313704) ·48077500

45254816

113137082) ·282268400

226274164

1131370844) ·5599423600

4525483376

11313708489) 107394022400

101823376401

113137084984) ··557064599900

452548339936

1131370849889) 10451625996400

10182337649001

11313708498982) ··26928834739900

22627416997964

·4301417741936

The

The Extraction of the Square-Root of the Number 32 by Approximation by means of the Expression $a + \frac{b}{2a}$ *given us for that purpose by Mr. Raphson.*

Art. 62. Let it now be required to find a near value of the square-root of the number 32, or 25 + 7, by means of the expression $a + \frac{b}{2a}$ found above, in art. 56, by Mr. Raphson's method of approximation.

Here in the 1st place we shall have $aa = 25$, and $b = 7$; and therefore a will be $= 5$, and $\frac{b}{2a}$ will be $(= \frac{7}{2 \times 5} = \frac{7}{10}) = 0.7$. Therefore $a + \frac{b}{2a}$ will be $(= 5 + 0.7) = 5.7$; or the square-root of $aa + b$, or 25 + 7, or 32, will be nearly equal to, but somewhat less than, 5.7. Q. E. I.

Secondly, since $\sqrt{32}$ is less than 5.7, let us suppose it to be nearly equal to 5.6, and let us raise 5.6 to it's square, in order to discover whether the said square will be greater, or less, than 32, and consequently whether 5.6 is greater, or less, than $\sqrt{32}$.

Now the square of 5.6 is 31.36; which is less than 32. Therefore 5.6 must be less than the square-root of 32.

We

We will therefore now take $a = 5.6$. And we ſhall then have $aa = 31.36$, and conſequently b $(= 32.00 - 31.36)$ 0.64, and $\frac{b}{2a}$ $(= \frac{0.64}{2 \times 5.6} = \frac{0.64}{11.2} =$ 0.057, and $a + \frac{b}{2a}$ $(= 5.6 + 0.057) = 5.657$. Therefore 5.657 will be a third near value of the ſquare-root of ($aa + b$, or $31.36 + 0.64$, or) the propoſed number 32. Q. E. I.

Now let 5.657 be raiſed to it's ſquare in order to diſcover whether the ſaid ſquare will be greater, or leſs, than 32, and conſequently whether 5.657 itſelf will be greater, or leſs, than the true value of $\sqrt{32}$.

Now $\overline{5.657}|^2$ is $= 32.001,649$; which is a little greater than 32. Therefore 5.657 is a little greater than the true value of $\sqrt{32}$. We will therefore ſuppoſe $\sqrt{32}$ to be between 5.657 and 5.656, and will raiſe 5.656 to it's ſquare, in order to diſcover whether the ſaid ſquare will be greater, or leſs, than 32, and conſequently whether 5.656 will be greater, or leſs, than $\sqrt{32}$.

Now the ſquare of 5.656 is $= 31.990,336$; which is leſs than 32; and conſequently 5.656 muſt be leſs than $\sqrt{32}$.

We will therefore now, in the 3d place, make $a = 5.656$; and we ſhall then have $aa = 31.990,336$, and b (=

b ($= 32.000,000 - aa = 32.000,000 - 31.990,336$) $= 0.009,664$. Therefore $\frac{b}{2a}$ will be ($= \frac{0.009,664}{2 \times 5.656} = \frac{0.009,664}{11.312}$) $= 0.000,854$, and confequently $a + \frac{b}{2a}$ will be ($= 5.656 + 0.000,854$) $= 5.656,854$. Therefore 5.656,854 will be a fourth near value of the fquare-root of 32. Q. E. I.

This fourth near value of the fquare-root of 32 is very nearly equal to, but fomewhat lefs than, it's true value. For the fquare of 5.656,854 is $= 31.999,997,177,316$, which is lefs than 32 by the very fmall number 0.000,002,822,684.

Note. If we had carried the divifion of b by $2a$, or of 0.009,664 by 11.312, to two figures more in the quotient, the faid quotient would have been 0.000,854,31, and confequently $a + \frac{b}{2a}$ would have been ($= 5.656, + 0.000,854,31$) $= 5.656,854,31$, which would have been greater than the true value of the fquare-root of 32, or $aa + b$, agreeably to what is fhewn above in art. 56. But by carrying the divifion to only the three fignificant figures 0.000,854, of the quotient, the value of $a + \frac{b}{2a}$ is kept under the true value of $\sqrt{32}$.

Since 5.656,854 is lefs than the true value of $\sqrt{32}$, we will, in order to obtain a nearer value of the faid fquare-

ſquare-root, take $a = 5.656,854$. And we ſhall then have $aa = 31.999,997,177,316$, and b ($= 32 - aa = 32.000,000,000,000 - 31.999,997,177,316$) $=$ $0.000,002,822,684$. Therefore $\frac{b}{2a}$ will be ($= \frac{0.000,002,822,684}{2 \times 5.656,854} = \frac{0.000,002,822,684}{11.313,708}$) $= 0.000,000,249,492$, and $a + \frac{b}{2a}$ will be ($= 5.656,854 + 0.000,000,249,492$) $= 5.656,854,249,492$; that is, the fifth near value of the ſquare-root of the number 32, obtained by this fourth proceſs of Mr. Raphſon's method of approximation, will be 5.656,854,249,492. Q. E. I.

All the figures of this number 5.656,854,249,492 are exact, and agree with the figures of the ſquare-root of 32 as obtained above in art. 61. by the common method of extracting it. But it appears to me that, from the neceſſity of ſquaring the ſeveral ſucceſſive values of a, in order to obtain the value of b, or $32 - aa$, and more eſpecially that of ſquaring the fourth value 5.656,854, (which conſiſts of ſeven figures,) and from the number of diviſions to be gone through in order to obtain the ſucceſſive values of $\frac{b}{2a}$, and likewiſe from the variety of new ſuppoſitions made in this method, and the reaſonings conſequent upon them, there is more time and trouble employed in obtaining theſe thirteen figures 5.656,854, 249,492 of the ſquare-root of the propoſed number 32 by this method of approximation than there is in obtaining them by the common method of extracting the ſquare-root in the manner exhibited above in art. 61.

I will

I will now proceed to extract the square-root of this same number 32 by means of the expression $a + \frac{b}{2a + \frac{b}{2a}}$, or $a + \frac{ab}{2aa + \frac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, given us by Monsieur de Lagny and Dr. Halley for the near value of the square-root of $aa + b$.

The Extraction of the Square-Root of the Number 32 by Approximation, by means of the Expression given us for that purpose by Monsieur de Lagny and Dr. Halley, and investigated above in Art. 57.

Art. 63. Of the three equivalent expressions $a + \frac{b}{2a + \frac{b}{2a}}$, $a + \frac{ab}{2aa + \frac{b}{2}}$, and $a + \frac{2ab}{4aa + b}$, found above in art. 57, for a near value of $\sqrt{aa + b}$, I take the first expression, $a + \frac{b}{2a + \frac{b}{2a}}$, to be the fittest for calculation; because it does not require our multiplying b into a, in order to obtain the quantity ab, which occurs in the numerators of the fractions $\frac{ab}{2aa + \frac{b}{2}}$ and $\frac{2ab}{4aa + b}$ in the two other expressions. I shall therefore here compute the near value of the square-

ſquare-root of 32, or 25 + 7, or $aa + b$, by means of the ſaid firſt expreſſion $a + \dfrac{b}{2a + \dfrac{b}{2a}}$.

Now, ſince aa is = 25, and b is = 7, we ſhall have $a = 5$, and $\dfrac{b}{2a}$ $\left(= \dfrac{7}{2 \times 5} = \dfrac{7}{10}\right) = 0.7$, and $2a + \dfrac{b}{2a}$ $(= 2 \times 5 + 0.7 = 10 + 0.7) = 10.7$, and conſequently $\dfrac{b}{2a + \dfrac{b}{2a}}$ $\left(= \dfrac{7}{10.7}\right) = 0.65$. Therefore $a + \dfrac{b}{2a + \dfrac{b}{2a}}$ will be $(= 5 + 0.65) = 5.65$; that is, the ſecond near value of the ſquare-root of 32, (obtained by means of the expreſſion $a + \dfrac{b}{2a + \dfrac{b}{2a}}$, after ſuppoſing it's firſt near value to be a, or 5,) will be 5.65; of which all the figures are exact. Q. E. I.

This value we know to be leſs than the truth. Therefore, to obtain a third near value of $\sqrt{32}$, we will make uſe of the ſame expreſſion as before, to wit, $a + \dfrac{b}{2a + \dfrac{b}{2a}}$. only making a (which before ſtood for 5, or the firſt near value of $\sqrt{32}$,) now ſtand for 5.65, or the ſecond near value $\sqrt{32}$, that has been juſt now obtained.

Now,

Now, if a is $= 5.65$, we shall have $aa = (\overline{5.65}|^2) = 31.9225$, and b $(= 32 - aa = 32.0000 - 31.9225) = 0.0775$. Therefore $\frac{b}{2a}$ will be $(= \frac{0.0775}{2 \times 5.65} = \frac{0.0775}{11.30}) = 0.006,858$, and $2a + \frac{b}{2a}$ will be $(= 11.30 + 0.006,858) = 11.306,858$, and $\frac{b}{2a + \frac{b}{2a}}$ will be $(= \frac{0.0775}{11.306,858}) = 0.006,854,24$. Therefore $a + \frac{b}{2a + \frac{b}{2a}}$ will be $(= 5.65 + 0.006,854,24) = 5.656,854,24$; that is, the third near value of the square-root of the number 32, which has been obtained by means of the expression $a + \frac{b}{2a + \frac{b}{2a}}$, will be $5.656,854,24$; which is exact in all it's nine figures.

Q. E. I.

We might now proceed to compute a fourth near value of this square-root by means of a third application of the expression $a + \frac{b}{2a + \frac{b}{2a}}$, by putting $a = 5.656,854,24$, and squaring the said number in order to obtain the value of aa, and then subtracting the said square from 32, in order to obtain the value of $N - aa$, or $32 - aa$, or b; and by, then, dividing b by $2a$, or $2 \times 5.656,854,24$, or by $11.313,708,48$, and adding the

the quotient, (continued to eight, or nine, figures,) to $2a$, or 11.313,708,48, and dividing b by the ſum thence ariſing, and adding the quotient of this laſt diviſion to a, or 5.656,854,24. And, if we were to do ſo, the near value of $\sqrt{32}$ thereby obtained would be exact to 27, or 26, figures. But the labour of the calculation would be very great; partly on account of the ſquaring of the long number 5.656,854,24, and partly on account of the diviſion of b by $2a + \frac{b}{2a}$, which, it is obvious, would be a moſt tedious operation, becauſe that diviſor would be a very long number, conſiſting of 17 or 18 figures. And therefore I conſider the further approximation to the value of the ſquare-root of 32 by means of this expreſſion $a + \cfrac{b}{2a + \cfrac{b}{2a}}$, as being perfectly inexpedient, and, in a manner, impracticable.

Art. 64. It remains that we ſhould inveſtigate the ſquare-root of the ſame number 32 by means of the third expreſſion $a + \frac{b}{2a} - \frac{bb}{8a^3}$, which co-incides with the three firſt terms of the infinite ſeries for expreſſing the value of $\sqrt{aa + b}$, or $\overline{aa + b}|^{\frac{1}{2}}$, derived from Sir Iſaac Newton's binomial theorem. Now this may be done as follows:

The Extraction of the Square-Root of the Number 32 by Approximation by means of the Trinomial Expression $a + \frac{b}{2a} - \frac{bb}{8a^3}$.

If a is at first taken equal to 5, (as before,) and consequently $aa = 25$, and b $(= 32 - aa = 32 - 25)$ $= 7$, we shall have $a + \frac{b}{2a} - \frac{bb}{8a^3}$ $(= 5 + \frac{7}{2 \times 5} - \frac{49}{8 \times 125} = 5 + \frac{7}{10} - \frac{49}{1000} = 5 + 0.7 - 0.049 = 5.700 - 0.049) = 5.651$; that is, 5.651, or (dropping the last figure 1,) 5.65, will be the second near value of $\sqrt{aa + b}$, or $\sqrt{25 + 7}$, or $\sqrt{32}$, that is obtained by this first application of the trinomial expression $a + \frac{b}{2a} - \frac{bb}{8a^3}$. Q. E. I.

Now let a be put $= 5.65$. And we shall then have aa $(= \overline{5.65}|^2) = 31.9225$, and b $(= 32.0000 - 31.9225) = 0.0775$. Therefore $\frac{b}{2a}$ will be $(= \frac{0.0775}{2 \times 5.65} = \frac{0.0775}{11.30}) = 0.006,858,40$, and $a + \frac{b}{2a}$ will be $(= 5.65 + 0.006,858,40) = 5.656,858,40$; and $\frac{bb}{4aa}$ will be $(= \frac{b}{2a} \times \frac{b}{2a} = \overline{0.006,858,40}|^2)$

=

$= 0.000,047,037,650,560,0$, and $\frac{bb}{8a^3}$ will be ($= \frac{bb}{2a \times 4aa} = \frac{0.000,047,037,650,560,0}{11.30}$) $= 0.000,004,16$ &c; and consequently $a + \frac{b}{2a} - \frac{bb}{8a^3}$ will be ($= 5.656,858,40 - 0.000,004,16$ &c) $= 5.656,854,24$; that is, the third near value of the square-root of 32, which is obtained by this second application of the trinomial expression $a + \frac{b}{2a} - \frac{bb}{8a^3}$, will be 5.656, 854,24; of which number all the nine figures are exact. Q. E. I.

If we were to carry this approximation one step further, by taking $a = 5.656,854,24$, and computing aa and $32 - aa$, or b, and the trinomial expression $a + \frac{b}{2a} - \frac{bb}{8a^3}$, or $a + \frac{b}{2a} - \frac{\frac{bb}{4aa}}{2a}$, we should obtain the value of the square-root of 32 exact to 26, or 27, figures. But the labour of computing the second and third terms $\frac{b}{2a}$ and $\frac{bb}{4aa}$, of this expression to that number of figures would be very great, though, I believe, not quite so great as that of computing $\frac{b}{2a + \frac{b}{2a}}$, (or the second term of the expression $a + \frac{b}{2a + \frac{b}{2a}}$ given by Monsieur de Lagny and

Dr. Halley for the ſame purpoſe,) to the ſame number of figures. I ſhould therefore conſider both theſe expreſſions in this caſe as unfit for practice, and ſhould think it would be much more adviſeable, if the ſquare-root of 32 was required to be found exactly to more than the nine figures 5.656,854,24 already diſcovered, to find the next nine figures of it's value either by computing Mr. Raphſon's much ſimpler and eaſier expreſſion $a + \frac{b}{2a}$, or by continuing the extraction of it by the common method to that extent.

Art. 65. And, indeed, from all theſe trials of theſe different expreſſions for approximating to the value of the ſquare-root of a given number N, or $aa + b$, I am inclined to conclude that, in performing the extractions of the ſquare-roots of given numbers which are neceſſary in Dr. Halley's method of reſolving high affected equations, it will almoſt always be found eaſier and more convenient to proceed by the common method of extracting them than to have recourſe to either of the three expreſſions above mentioned, not excepting even Mr. Raphſon's ſimple expreſſion $a + \frac{b}{2a}$.

End of the Obſervations on the Extraction of the Square-Root of a given Number by the Methods of Approximation.

A ME-

A METHOD OF DISCOVERING WHETHER THE THREE EQUATIONS THAT HAVE BEEN RESOLVED IN THE FOREGOING ARTICLES HAVE ANY OTHER ROOTS BESIDES THOSE ABOVE INVESTIGATED.

Of the Number of Roots in the Cubick Equation $x^3 - 17x^2 + 54x = 350$.

Art. 66. THE firſt of the three equations that have been reſolved in the foregoing articles is the cubick equation $x^3 - 17x^2 + 54x = 350$, of which we have found one root to be $= 14.954,068,61$. Now, as there are two changes of the ſigns + and — prefixed to the terms that form the left-hand ſide of this equation, it may be ſuſpected that there may be either one or two more roots to it, or values of x that, being ſubſtituted inſtead of x in the trinomial quantity $x^3 - 17x^2 + 54x$, will make the ſaid quantity be equal to 350, or the abſolute term of the ſaid equation. This therefore is a matter which ought to be inquired-into and determined, before the reſolution of the equation can be conſidered as quite compleat.

Art. 67. Now, if this equation has another root, that root muſt be either greater or leſs than 14.954,068,61, or (omitting all the figures of this root except the three firſt, 14.9, in order to leſſen the labour of the following calculations,) leſs than 14.9. Let it firſt be ſuppoſed to be greater than 14.9, and to be called c; and let b be put $= 14.9$.

Then

Then we ſhall have $b^3 - 17b^2 + 54b = 350$, and alſo $c^3 - 17c^2 + 54c = 350$; and conſequently $c^3 - 17c^2 + 54c$ will be $= b^3 - 17b^2 + 54b$. Therefore (adding $17c^2$ to both ſides) we ſhall have $c^3 + 54c = b^3 + 17c^2 - 17b^2 + 54b$, and (ſubtracting $b^3 + 54b$ from both ſides) $c^3 - b^3 + 54c - 54b = 17c^2 - 17b^2$, or $c^3 - b^3 + 54 \times \overline{c - b} = 17 \times \overline{c^2 - b^2}$. Therefore $\frac{c^3 - b^3}{c - b} + 54 \times \frac{c - b}{c - b}$ will be $= 17 \times \frac{\overline{c^2 - b^2}}{c - b}$, or $c^2 + cb + b^2 + 54$ will be $= 17 \times \overline{c + b} = 17c + 17b$. Therefore (ſubſtituting 14.9 inſtead of b,) $c^2 + 14.9 \times c + \overline{14.9}|^2 + 54$ will be $= 17c + 17 \times 14.9$, that is, $c^2 + 14.9 \times c + 222.01 + 54$ will be $= 17c + 253.3$. Therefore (ſubtracting 222.01 from both ſides,) we ſhall have $c^2 + 14.9 \times c + 54 = 17c + 31.29$, and (ſubtracting 31.29 from both ſides) $c^2 + 14.9 \times c + 22.71 = 17c$, and (ſubtracting $14.9c$ from both ſides,) $c^2 + 22.71 = 2.1 \times c$; that is, the ſum of c^2 and 22.71 is equal to $2.1 \times c$. Therefore c^2 alone muſt be leſs than $2.1 \times c$, and conſequently (dividing both ſides by c,) c muſt be leſs than 2.1. But c was ſuppoſed to be greater than b, or 14.9, which is much greater than 2.1. Therefore c or the ſuppoſed ſecond value of x in the equation $x^3 - 17x^2 - 54x = 350$ will be both greater and leſs than 2.1; which is impoſſible. And this impoſſible concluſion followed from the ſuppoſition that the equation $x^3 - 17x^2 + 54x = 350$ had another root c that was greater than b,

or

or 14.9. Therefore that ſuppoſition was a falſe ſuppoſition, and the ſaid equation cannot have any other root that is greater than 14.9.

Art. 68. It remains that we inquire whether the ſaid equation cannot have another root beſides b, or 14.9, that ſhall be leſs than 14.9.

Now, if this be poſſible, let the ſaid root, that is leſs than b, or 14.9, be called a.

Then we ſhall have $b^3 - 17b^2 + 54b = 350$, and $a^3 - 17a^2 + 54a = 350$, and conſequently $b^3 - 17b^2 + 54b = a^3 - 17a^2 + 54a$. Therefore, (adding $17b^2$ to both ſides,) we ſhall have $b^3 + 54b = a^3 + 17b^2 - 17a^2 + 54a$, and (ſubtracting $a^3 + 54a$ from both ſides,) $b^3 - a^3 + 54b - 54a = 17b^2 - 17a^2$, or $b^3 - a^3 + 54 \times \overline{b-a} = 17 \times \overline{b^2 - a^2}$. Therefore $\dfrac{b^3 - a^3}{b-a} + 54 \times \dfrac{b-a}{b-a}$ will be $= 17 \times \dfrac{b^2 - a^2}{b-a}$; that is, $b^2 + ba + aa + 54$ will be $= 17 \times \overline{b+a} = 17b + 17a$; or (ſubſtituting 14.9 inſtead of b in this equation) $\overline{14.9}|^2 + 14.9 \times a + aa + 54$ will be $= 17 \times 14.9 + 17a$, or $222.01 + 14.9 \times a + aa + 54$ will be $= 253.30 + 17a$. Therefore (ſubtracting 222.01 from both ſides,) we ſhall have $14.9 \times a + aa + 54 = 31.29 + 17a$, and (ſubtracting 31.29 from both ſides,) $14.9 \times a + aa + 22.71 = 17a$, and (ſubtracting $14.9 \times a$ from both ſides,) $aa + 22.71 = 2.1 \times a$; and, laſtly, (ſubtracting aa from both ſides,) $2.1 \times a - aa = 22.71$, or $\overline{2.1 - a}| \times a =$

22.71;

22.71; that is, the product of the multiplication of 2.1 — a into a will be equal to 22.71. But the greatest possible product of the multiplication of 2.1 — a into a is the square of half 2.1, or the square of 1.05, which is 1.1025. Therefore it is impossible that the binomial quantity $2.1\,a - aa$ should be equal to 22.71. And this impossible conclusion has been regularly deduced from the supposition that the equation $x^3 - 17x^2 + 54x = 350$ had another root besides 14.9, that was less than 14.9. Therefore that supposition was a false supposition, and the said equation $x^3 - 17x^2 + 54x = 350$ cannot have any root besides 14.9, that is less than 14.9.

But it was shewn before, that the said equation cannot have any root besides 14.9 that is greater than 14.9.

Therefore the said equation cannot have any other root whatsoever besides 14.9, or 14.954,068,61; and therefore this number will be it's only root. Q. E. I.

Another Method of proving that the Equation $x^3 - 17x^2 + 54x = 350$ *has no other Root but the Root* 14.954,068,61, *that has been already found.*

Art. 69. In the two foregoing articles it has been shewn that the equation $x^3 - 17x^2 + 54x = 350$ cannot have any other root besides 14.954,068,61, or 14.9, by proving that, if it were supposed to have any other root, either greater or less than 14.9, an impossible conclusion would follow from such supposition. This indirect way of proving a proposition is not quite so satisfactory, or, at least, so pleasing to the mind, as a direct

direct proof. I will therefore now endeavour to prove the same thing in a direct manner, by shewing, 1st, that, if we suppose x to increase from 14.9, or b, to any greater quantity c, which exceeds 14.9, or b, by the difference d, the sum of the increments received by the trinomial quantity $x^3 - 17x^2 + 54x$, or $b^3 - 17b^2 + 54b$, during the increase of x from b to c, or $b + d$, will be greater than the sum of the decrements which it will suffer, or the sum of the quantities that will be subtracted from it, and consequently that the said trinomial quantity $x^3 - 17x^2 + 54x$ (which will then be equal to $c^3 - 17c^2 + 54c$, or $\overline{b+d}|^3 - 17 \times \overline{b+d}|^2 + 54 \times \overline{b+d}$) will, upon the whole, be greater than it was before, or than the absolute term 350, and consequently that c, or $b + d$, will not be a root of the equation $x^3 - 17x^2 + 54x = 350$; and by shewing, 2ndly, that, if we suppose x to decrease from 14.9, or b, to any lesser quantity a, which falls short of b by the difference d, the sum of the increments received by the trinomial quantity $x^3 - 17x^2 + 54x$, or $b^3 - 17b^2 + 54b$, during the decrease of x from b to a, or $b - d$, will be less than the sum of the decrements it will suffer, or the sum of the quantities that will be subtracted from it, and consequently that the said trinomial quantity $x^3 - 17x^2 + 54x$ (which will then be equal to $a^3 - 17a^2 + 54a$, or $\overline{b-d}|^3 - 17 \times \overline{b-d}|^2 + 17 \times \overline{b-d}$,) will, upon the whole, be less than it was before, or than the absolute term 350, and consequently that a, or $b - d$, will not be a root of

of the equation $x^3 - 17x^2 + 54x = 350$. These things may be proved in the manner following:

Art. 70. If x is supposed to increase from b, or $14.\dot{9}$, to c, or $b + d$, the trinomial quantity $x^3 - 17x^2 + 54x$, will be changed from $b^3 - 17b^2 + 54b$ to $c^3 - 17c^2 + 54c$, or $\overline{b+d}|^3 - 17 \times \overline{b+d}|^2 + 54 \times \overline{b+d}$, or $b^3 + 3b^2d + 3bd^2 + d^3 - 17 \times \overline{b^2 + 2bd + d^2} + 54 \times \overline{b+d}$, or $b^3 + 3b^2d + 3bd^2 + d^3 - 17b^2 - 34bd - 17d^2 + 54b + 54d$, or

$$\left\{\begin{array}{l} \quad b^3 + 3b^2d + 3bd^2 + d^3 \\ -17b^2 - 34bd - 17d^2 \\ +54b + 54d \end{array}\right\}; \text{ which}$$

contains the trinomial quantity $b^3 - 17b^2 + 54b$ (which is equal to 350) together with the four quantities $3b^2d$, $3bd^2$, d^3, and $54d$, which are marked with the sign $+$, or are added to $b^3 - 17b^2 + 54d$, and consequently tend to increase it, and the two quantities $34bd$ and $17d^2$, which are marked with the sign $-$, or are subtracted from $b^3 - 17b^2 + 54b$, and consequently tend to diminish it. We must therefore demonstrate that the sum of the four quantities $3b^2d$, $3bd^2$, d^3, and $54d$ will be greater than the sum of the two quantities $34bd$ and $17d^2$.

Art. 71. Now the quadrinomial quantity $3b^2d + 3bd^2 + d^3 + 54d$ is $(= 3 \times \overline{14.9}|^2 \times d + 3 \times$

14.9

$14.9 \times d^2 + d^3 + 54d = 3 \times 222.01 \times d + 44.7 \times d^2 + d^3 + 54d = 666.03 \times d + 44.7 \times d^2 + d^3 + 54d) = 720.03 \times d + 44.7 \times d^2 + d^3$; and the binomial quantity $34bd + 17d^2$ is $(= 34 \times 14.9 \times d + 17d^2) = 506.6 \times d + 17d^2$.

But $720.03 \times d$ is greater than $506.6 \times d$, and $44.7 \times d^2$ is greater than $17d^2$. Therefore the binomial quantity $720.03 \times d + 44.7 \times d^2$ will be greater than the binomial quantity $506.6 \times d + 17d^2$. Therefore, *à fortiori*, the trinomial quantity $720.03 \times d + 44.7 \times d^2 + d^3$ will be greater than the binomial quantity $506.6 \times d + 17d$. Therefore the quadrinomial quantity $3b^2d + 3bd^2 + d^3 + 54d$ (which is equal to the trinomial quantity $720.03 \times d + 44.7 \times d^2 + d^3$,) will be greater than the binomial quantity $506.6 \times d + 17d$, and confequently than it's equal, the binomial quantity $34bd + 17d^2$. Therefore the trinomial quantity $\overline{b+d}|^3 - 17 \times \overline{b+d}|^2 + 54 \times b + d$ will be greater than the trinomial quantity $b^3 - 17b^2 + 54b$, or the abfolute term 350; and confequently c, or $b + d$, cannot be a root of the propofed equation $x^3 - 17x^2 + 54x = 350$.

Q. E. D.

Art. 72. Secondly, if x is fuppofed to decreafe from b, or 14.9, to a, or $b - d$, the trinomial quantity $x^3 - 17x^2 + 54x$ will be changed from $b^3 - 17b^2 + 54b$ to $a^3 - 17a^2 + 54a$, or $\overline{b-d}|^3 - 17 \times \overline{b-d}|^2 + 54 \times \overline{b-d}$, or $b^3 - 3b^2d + 3bd^2 - d^3 - 17 \times$

b^2

$\overline{b^2 - 2bd + d^2} + 54 \times \overline{b - d}$, or $b^3 - 3b^2d + 3bd^2 - d^3 - 17b^2 + 34bd - 17d^2 + 54b - 54d$, or

$$\left\{ \begin{array}{l} b^3 - 3b^2d + 3bd^2 - d^3 \\ -17b^2 + 34bd - 17d^2 \\ +54b - 54d \end{array} \right\};$$ which

contains the trinomial quantity $b^3 - 17b^2 + 54b$ (which is equal to 350,) together with the four quantities $3b^2d$, d^3, $17d^2$, and $54d$, which are marked with the fign $-$, or fubtracted from the quantity $b^3 - 17b^2 + 54b$, and confequently tend to diminifh it, and the two quantities $3bd^2$ and $34bd$, which are marked with the fign $+$, or are added to the quantity $b^3 - 17b^2 + 54b$, and confequently tend to increafe it. We muft therefore demonftrate that the fum of the four quantities $3b^2d$, d^3, $17d^2$, and $54d$, which are marked with the fign $-$, is greater than the fum of the two quantities $3bd^2$ and $34bd$, which are marked with the fign $+$.

Art. 73. Now the quadrinomial quantity $3b^2d + d^3 + 17d^2 + 54d$ is $(= 3 \times \overline{14.9}|^2 \times d + d^3 + 17d^2 + 54d = 3 \times 222.01 \times d + d^3 + 17d^2 + 54d = 666.03 \times d + d^3 + 17d^2 + 54d) = 720.03 \times d + 17d^2 + d^3$; and the binomial quantity $3bd^2 + 34bd$ is $= 3 \times 14.9 \times d^2 + 34 \times 14.9 \times d) = 44.7 \times d^2 + 506.6 \times d$. Therefore, if we can fhew that the trinomial quantity $720.03 \times d + 17d^2 + d^3$ is greater than the binomial quantity $44.7 \times d^2 +$

506.6

$506.6 \times d$, or $506.6 \times d + 44.7 \times d^2$, it will follow that the quadrinomial quantity $3b^2d + d^3 + 17d^2 + 54$ will be greater than the binomial quantity $3bd^2 + 34bd$, and consequently that the trinomial quantity $a^3 - 17a^2 + 54a$, or $\overline{b-d}|^3 - 17 \times \overline{b-d}|^2 + 54 \times \overline{b-d}$, will be lefs than the trinomial quantity $b^3 - 17b^2 + 54b$, or than the abfolute term 350 of the propofed equation $x^3 - 17x^2 + 54x = 350$, and confequently that a, or $b - d$, cannot be a root of the faid equation.

Art. 74. Now that the trinomial quantity $720.03 \times d + 17d^2 + d^3$ muft always be greater than the binomial quantity $506.6 \times d + 44.7 \times d^2$ may be demonftrated as follows:

The trinomial quantity $720.03 \times d + 17d^2 + d^3$ will be greater than the binomial quantity $506.6 \times d + 44.7 \times d^2$ if the quotient of the divifion of the former quantity by d is greater than the quotient of the divifion of the latter quantity by d, that is, if the trinomial quantity $720.03 + 17d + d^2$ is greater than the binomial quantity $506.6 + 44.7 \times d$. We muft therefore demonftrate that the trinomial quantity $720.03 + 17d + dd$ is greater than the binomial quantity $506.6 + 44.7 \times d$.

Now the trinomial quantity $720.03 + 17d + dd$ will be greater than the binomial quantity $506.6 + 44.7 \times d$, if $720.03 - 506.60 + 17d + dd$ is greater than

than $44.7 \times d$, or if $720.03 - 506.60 + dd$ is greater than $44.7 \times d - 17d$, or if $213.43 + dd$ is greater than $27.7 \times d$, or if 213.43 is greater than $27.7 \times d - dd$, or than $\overline{27.7 - d} \times d$. But the greatest possible magnitude of $\overline{27.7 - d} \times d$ is the square of half 27.7, or the square of 13.85, which is less than the square of 14, or than 196. Therefore 213.43 (which is greater than 196,) must be greater than $\overline{27.7 - d} \times d$, or than $27.7 \times d - dd$; and consequently the trinomial quantity $720.03 + 17d + dd$ will be greater than the binomial quantity $506.6 + 44.7 \times d$, and the trinomial quantity $720.03 \times d + 17d^2 + d^3$ will be greater than the binomial quantity $506.6 \times d + 44.7 + dd$.

Q. E. D.

It appears therefore that neither $b + d$, nor $b - d$, that is, that no quantity either greater or less than b, or 14.9, or $14.954,068,61$, can be a root of the proposed equation $x^3 - 17x^2 + 54x = 350$.

Of the Number of Roots in the Biquadratick Equation $x^4 - 3xx + 75x = 10,000$.

Art. 75. We will next examine the second equation above-resolved, to wit, the biquadratick equation $x^4 - 3x^2 + 75x = 10,000$, of which we have found one root to be $= 9.886,002,70$, and will proceed to inquire whether the said equation has, or has not, any other root, or whether there is any other quantity besides $9.886,002,70$, which, being substituted instead of x in the trinomial quantity $x^4 - 3x^2 + 75x$, will make that quantity be equal to the absolute term $10,000$.

Now

Now this inquiry may be made with respect to this equation with great ease and brevity in the following manner:

If x is supposed to increase from 0 *ad infinitum*, it is evident that, while x is less than $\sqrt{3}$, and xx is consequently less than 3, x^4 will be less than $3xx$, and consequently the whole trinomial quantity $x^4 - 3xx + 75x$ must be less than $75x$. But, while x is less than $\sqrt{3}$, $75x$ will be less than $75 \times \sqrt{3}$, or 75×1.732, &c, or 129.9. Therefore, while x is less than $\sqrt{3}$, the whole trinomial quantity, (which is less than $75x$,) will, *à fortiori*, be less than 129.9, and therefore will be much less than 10,000, or the absolute term of the equation $x^4 - 3xx + 75x = 10,000$. Therefore it is plain that no quantity less than 1.732, or $\sqrt{3}$, can be a root of this equation. And, when x is greater than $\sqrt{3}$, and xx is greater than 3, it is evident that x^4 must be greater than $3xx$, and that, while xx increases from 3 *ad infinitum*, the compound quantity $x^4 - 3xx$ (which is $= xx \times \overline{xx - 3}$) will increase continually at the same time from 0 *ad infinitum*, and the third term $75x$ will likewise increase continually from $75 \times \sqrt{3}$ *ad infinitum*. Therefore the sum of these two quantities $x^4 - 3xx$ and $75x$, that is, the trinomial quantity $x^4 - 3xx + 75x$, will likewise increase from ($0 + 75 \times \sqrt{3}$, or from) $75 \times \sqrt{3}$ *ad infinitum* without ever decreasing, and therefore will once, and but once, become equal to any proposed quantity whatsoever that is greater than $75 \times \sqrt{3}$, and consequently will once, and but once, become equal to 10,000, or the absolute term

term of the equation $x^4 - 3xx + 75x = 10{,}000$; or, in other words, that equation will have one, and but one, root. Q. E. I.

This eafy method of proving that this equation can have only one root was fuggefted to me by the ingenious *Mr. William Frend*, Fellow of Jefus College, Cambridge, and author of a valuable Treatife on Algebra in one volume, octavo, intitled *Principles of Algebra*, in which he has explained the fubject in a very clear and fcientifick manner, and without any mention of the obfcure and ufelefs doctrine of negative quantities, or quantities lefs than nothing, or quantities obtained by fubtracting a greater quantity from a leffer.

Of the Number of Roots in the Equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$.

Art. 76. We will laftly examine the third equation above-refolved, to wit, the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, and inquire, whether, or not, it has any other roots befides the root above found, to wit, 12.756,441,794,480,744,022.

Now let b be put for the root 12.756, &c that is already found, or, rather, for the whole number 13, (which is nearly equal to it,) in order to facilitate the calculations which we fhall have occafion to make in the courfe of our inquiry. And let us then fuppofe x to increafe from 13, or b, to fome greater quantity which we will call c, and which fhall exceed the known root 13, or b, by the difference d.

Then

Then it is evident that, when x is increased from 13, or b, to c, or $b+d$, the quadrinomial quantity $14,937\,b - 1998\,b^2 + 80\,b^3 - b^4$ (which is equal to the absolute term 5000) will be changed into $14,937\,c - 1998\,c^2 + 80\,c^3 - c^4$, or into $14,937 \times \overline{b+d} - 1998 \times \overline{b+d}|^2 + 80 \times \overline{b+d}|^3 - \overline{b+d}|^4$, that is, into $14,937 \times \overline{b+d} - 1998 \times \overline{b^2 + 2bd + d^2} + 80 \times \overline{b^3 + 3b^2d + 3bd^2 + c^3} - \overline{b^4 + 4b^3d + 6b^2d^2 + 4bd^3 + d^4}$, or into the compound quantity

$$\left\{\begin{array}{lllll} \quad 14,937\,b & + \ 14,937\,d & & & \\ -\ 1998\,b^2 & -\ 3996\,bd & -\ 1998\,d^2 & & \\ +\quad 80\,b^3 & +\ 240\,b^2d & +\ 240\,bd^2 & +\ 80\,d^3 & \\ -\quad b^4 & -\ 4\,b^3d & -\ 6\,b^2d^2 & -\ 4\,bd^3 & -\ d^4; \end{array}\right\}$$

which contains the quadrinomial quantity $14,937\,b - 1998\,b^2 + 80\,b^3 - b^4$, (which is equal to 5000,) together with the four quantities $14,937\,d$, $240\,b^2d$, $240\,bd^2$, and $80\,d^3$, which are marked with the sign $+$, or are added to the quadrinomial quantity $14,937\,b - 1998\,b^2 + 80\,b^3 - b^4$, and consequently tend to increase it, and the six quantities $3996\,bd$, $1998\,d^2$, $4\,b^3d$, $6\,b^2d^2$, $4\,bd^3$, and d^4, which are marked with the sign $-$, or are subtracted from the said quadrinomial quantity, and therefore tend to diminish it. We must therefore compare the sum of the four quantities $14,937\,d$, $240\,b^2d$, $240\,bd^2$, and $80\,d^3$ with the sum of the six quantities $3996\,bd$, $1998\,d^2$, $4\,b^3d$, $6\,b^2d^2$,

$4bd^3$, and d^4, and inquire whether the former ſum can ever be exactly equal to the latter ſum (in which caſe $b+d$ would be another root of the equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$,) or whether it muſt not be always either greater or leſs than the ſaid latter ſum, in which caſe it will be impoſſible for c, or $b+d$, or any quantity greater than b, or 13, to be a root of the ſaid equation.

Art. 77. Now, ſince b is $= 13$, we ſhall have b^2 $(= \overline{13}^2) = 169$, and b^3 $(= \overline{13}^3) = 2197$, and conſequently $240b^2d$ $(= 240 \times 169 \times d) = 40,560d$, and $240bd^2$ $(= 240 \times 13 \times d^2) = 3120d^2$, and $3996bd$ $(= 3996 \times 13 \times d) = 51,948d$, and $4b^3d$ $(= 4 \times 2197 \times d) = 8788d$, and $6b^2d^2$ $(= 6 \times 169 \times d^2) = 1014d^2$, and $4bd^3$ $(= 4 \times 13 \times d^3) = 52d^3$. Therefore the four quantities $14,937d + 240b^2d + 240bd^2 + 80d^3$ will be $(= 14,937d + 40,560d + 3120d^2 + 80d^3) = 55,497d + 3120d^2 + 80d^3$; and the ſix quantities $3996bd + 1998d^2 + 4b^3d + 6b^2d^2 + 4bd^3 + d^4$ will be $(= 51,948d + 1998d^2 + 8788d + 1014d^2 + 52d^3 + d^4) = 60,736d + 3012d^2 + 52d^3 + d^4$. We muſt therefore compare the trinomial quantity $55,497d + 3120d^2 + 80d^3$ with the quadrinomial quantity $60,736d + 3012d^2 + 52d^3 + d^4$, in order to diſcover whether it is poſſible that they ſhould be equal to each other.

Now the trinomial quantity $55,497d + 3120d^2 + 80d^3$ will be equal to the quadrinomial quantity

$60,736d$

$60,736d + 3012d^2 + 52d^3 + d^4$ if the binomial quantity $3120d^2 + 80d^3$ is $=$ to $60,736d - 55,497d + 3012d^2 + 52d^3 + d^4$, or to $5239d + 3012d^2 + 52d^3 + d^4$, or if the quantity $3120d^2 - 3012d^2 + 80d^3$, or the binomial quantity $108d^2 + 80d^3$ is $=$ to the trinomial quantity $5239d + 52d^3 + d^4$, or if the quantity $108d^2 + 80d^3 - 52d^3$, or the binomial quantity $108d^2 + 28d^3$, is $=$ the binomial quantity $5239d + d^4$, or (dividing all the terms by d) if the binomial quantity $108d + 28dd$ is equal to the binomial quantity $5239 + d^3$, or if the trinomial quantity $108d + 28dd - d^3$ is equal to the single quantity 5239.

But the trinomial quantity $108d + 28dd - d^3$ will, in the course of the increase of x from b, or 13, *ad infinitum*, or of the increase of d (or the excess of c, or $b + d$, above b,) from 0 *ad infinitum* become equal to 5239; as will appear from the following considerations:

Art. 78. While d is increasing from 0 to 1, d^3 will be less than d; but when d is greater than 1, d^3 will be greater than d; and, while d increases from 1 *ad infinitum*, d^3 will become equal, first, to $108d$, and then to $108d + 28dd$, and afterwards will be greater than $108d + 28dd$, and will so continue, to whatever greater magnitude it may be supposed to increase. Now, when d^3 is become equal to $108d + 28dd$, the trinomial quantity $108d + 28dd - d^3$ will be $= 0$. Therefore, while d increases from 0 to the value which it has when d^3 is equal to $108d + 28dd$, the trinomial quan-

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tity $108d + 28dd - d^3$ will, firſt, increaſe from o to a certain magnitude, and then decreaſe from that magnitude to o. And conſequently, if that greateſt magnitude of the trinomial quantity $108d + 28dd - d^3$, during the ſaid increaſe of d from o to the value which it has when d^3 is equal to $108d + 28dd$, is greater than 5239, the ſaid trinomial quantity $108d + 28dd - d^3$ will, at two different inſtants of time, the one before it attains it's greateſt magnitude, and the other after it has attained it, and is decreaſing to o, be equal to 5239; and conſequently there will be two values of d which, being added to b, will make the compound quantity $14{,}937 \times \overline{b + d} - 1998 \times \overline{b + d}|^2 + 80 \times \overline{b + d}|^3 - \overline{b + d}|^4$ be equal to $14{,}937b - 1998b^2 + 80b^3 - b^4$, or to the abſolute term 5000 of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, or there will be two values of $b + d$, or two quantities greater than b, or 13, that will be roots of that equation. Now it will be found upon examination that the greateſt magnitude which the ſaid trinomial quantity $108d + 28dd - d^3$ will attain during the ſaid increaſe of d from o to the value which it has when d^3 is equal to $108d + 28dd$, will be greater than 5239, and therefore there will be two values of $b + d$, or two quantities greater than b, or 13, that will be roots of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Theſe things may be ſhewn in the manner following:

Art. 79. In the firſt place we will inquire into the magnitude of d when d^3 is equal to $108d + 28dd$. Now

Now this value of d may be determined by refolving a quadratick equation. For when d^3 is $= 108\,d + 28\,dd$, we fhall (by dividing all the terms by d) have $dd = 108 + 28\,d$, and confequently $dd - 28\,d = 108$. Therefore $dd - 28\,d + \overline{14}|^2$ will be $(108 + \overline{14}|^2 = 108 + 196) = 304$, and $d - 14$ will be $(= \sqrt{304}) = 17.4$, and d will be $(= 17.4 + 14) = 31.4$, or 31.4 will be the magnitude of d when d^3 is equal to $108\,d + 28\,dd$. Q. E. I.

Therefore, while d increafes from 0 to 31.4, the trinomial quantity $108\,d + 28\,dd - d^3$ will, firft, increafe from 0 to a certain magnitude, and then will decreafe from that magnitude and become equal to 0 again.

We will next inquire what that magnitude is, to which the faid trinomial quantity $108\,d + 28\,dd - d^3$ will have increafed, and from which it will afterwards have decreafed to 0, during the faid increafe of d from 0 to 31.4, in order to difcover whether that magnitude is greater, or lefs, than 5239. For, if it is greater than 5239, it is evident that the trinomial quantity $108\,d + 28\,dd - d^3$ will, at two different inftants of time during the increafe of d from 0 to 31.4, namely, once before the faid trinomial quantity has attained it's greateft magnitude, and a fecond time after it has attained it's greateft magnitude, and while it is decreafing from that magnitude to 0, become equal to 5239.

Now let us fuppofe $\dot{d}$, or the letter d with a point placed over it, to denote fome extremely fmall part of

31.4, (or of the value of d when the trinomial quantity $108d + 28dd - d^3$ is become a fecond time equal to o,) as, for example, the 10,000,000,000th, or ten-thoufand millionth, part of it. Then it is evident that while d, in it's increafe from o to 31.4, receives one of thefe fmall increments denoted by $\dot{d}$, the fquare of d will receive the fmall increment $2d\dot{d}$, and d^3, or the cube of d, will receive the fmall increment $3d^2\dot{d}$, and the quantities $108d$ and $28dd$ will receive the fmall increments $108 \times \dot{d}$ and $28 \times 2d\dot{d}$, or $56d\dot{d}$. Therefore the increment of the binomial quantity $108d + 28dd$ will be $108\dot{d} + 56d\dot{d}$.

Now this increment $108\dot{d} + 56d\dot{d}$ will at firft be greater than $3d^2\dot{d}$, the contemporary increment of d^3; becaufe $108 + 56d$ will at firft be much greater than $3dd$. But, as d increafes, the ratio of $108 + 56d$ to $3dd$ (which at firft is a very great ratio of majority,) will for fome time continually grow lefs and lefs and approach to a ratio of equality, and will then become a ratio of equality, and afterwards will become a ratio of minority. And when this ratio is a ratio of equality, or $108 + 56d$ is $= 3dd$, we fhall have $\frac{108}{3} + \frac{56d}{3} = dd$, or $36 + 18.66 \times d = dd$, and confequently $dd - 18.66 \times d = 36$, and $dd - 18.66 \times d + \overline{9.33}|^2$ $(= 36 + \overline{9.33}|^2 = 36 + 87.0489) = 123.0489$, and $d - 9.33$ $(= \sqrt{123.0489}) = 11.09$, and d $(= 11.09 + 9.33) = 20.42$. Now, fo long as the increment,

ment, $108\dot{d} + 56 d\dot{d}$, of the binomial quantity $108 d + 28 dd$ is greater than $3 d^2 \dot{d}$, the contemporary increment of d^3, it is evident that the trinomial quantity $108 d + 28 dd - d^3$, or the excess of the binomial quantity $108 d + 28 dd$ above d^3, will increase; but, when the increment of d^3 becomes greater than the increment of $108 d + 28 dd$, the said trinomial quantity will decrease. And consequently the greatest magnitude of the said trinomial quantity $108 d + 28 dd - d^3$ will be that which it has when the said increments are equal to each other, or when d is $= 20.42$.

But, when d is $= 20.42$, we shall have $108 d$ ($= 108 \times 20.42$) $= 2205.36$, and $28 dd$ ($= 28 \times \overline{20.42}|^2 = 28 \times 416.9764$) $= 11,675.3392$, and d^3 ($= dd \times d = 416.9764 \times 20.42$) $= 8514.658,088$, and consequently the trinomial quantity $108 d + 28 dd - d^3$ will be ($= 2205.36 + 11,675.3392 - 8514.658,088 = 13,880.6992 - 8514.658,088$) $= 5366.041,112$. And consequently this quantity $5366.041,112$ will be the greatest magnitude of the said trinomial quantity.

Now this quantity $5366.041,112$, or (dropping the fractional part of it, as inconsiderable) 5366, is a little greater than 5239. Therefore, while d increases from 0 to 31.4, the trinomial quantity $108 d + 28 dd - d^3$ will, at two different times, become equal to the quantity 5239, namely, once a little before it attains it's greatest magnitude 5366, and a second time a little after it has attained it's said greatest magnitude, and is decreasing. And therefore there will be two values of d, the one a little

less

lefs than 20.42, and the other a little greater than 20.42, that will make the quantities $b + d$ be roots of the propofed equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$. So that this equation will have two roots that will be greater than 13, of which the one will be fomewhat lefs than $b + 20.42$, or $13 + 20.42$, or 33.42, and the other will be fomewhat greater than 33.42.

Q. E. I.

An Inquiry, whether the Equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$ *may not have a fourth Root lefs than the Root* 12.756,441, *&c. already found.*

Art. 80. We will now inquire whether the faid equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$ may not have a fourth root that fhall be lefs than the root 12.756,441,794,480,744,022, which has been already inveftigated, as well as the two roots that are greater than that root, and not very different from the number 33.42.

We will therefore fuppofe x to decreafe from 12.756, &c, or 13, or b, to fome leffer quantity which we will call a, and which fhall fall fhort of b, or 13, by a difference called d.

Then it is evident that, when x has decreafed from b, or 13, to a, or $b - d$, the quadrinomial quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$, will be changed into the quadrinomial quantity $14{,}937\,a - 1998\,a^2 + 80\,a^3 - a^4$, or into the quadrinomial quantity $14{,}937 \times \overline{b - d} - 1998 \times \overline{b - d}|^2 + 80 \times \overline{b - d}|^3 - \overline{b - d}|^4$, or into the quadrinomial quantity $14{,}937 \times \overline{b - d} - 1998 \times$

b^2

$\overline{b^2 - 2bd + d^2} + 80 \times \overline{b^3 - 3b^2d + 3bd^2 - d^3}) - \overline{(b^4 - 4b^3d + 6b^2d^2 - 4bd^3 - d^4}$, or into the multinomial quantity

$$\left\{\begin{array}{l} 14,937\,b - 14,937\,d \\ - 1998\,b^2 + 3996\,bd - 1998\,d^2 \\ + \quad 80\,b^3 - 240\,b^2d + 240\,bd^2 - 80\,d^3 \\ - \qquad b^4 + \quad 4\,b^3d - 6\,b^2d^2 + 4bd^3 - d^4; \end{array}\right\}$$

which contains the quadrinomial quantity $14,937\,b - 1998\,b^2 + 80\,b^3 - b^4$, (which is equal to 5000,) together with the six quantities $14,937\,d$, $1998\,d^2$, $240\,b^2d$, $80\,d^3$, $6\,b^2d$, and d^4, which are marked with the sign —, or are subtracted from the quantity $14,937\,b - 1998\,b^2 + 80\,b^3 - b^4$, and consequently tend to diminish it, and the four quantities $3996\,bd$, $240\,bd^2$, $4\,b^3d$, and $4\,bd^3$, which are marked with the sign +, or are added to the quantity $14,937\,b - 1998\,b^2 + 80\,b^3 - b^4$, and consequently tend to increase it. We must therefore compare the sum of the six quantities $14,937\,d$, $1998\,d^2$, $240\,b^2d$, $80\,d^3$, $6\,b^2d^2$, and d^4 with the sum of the four quantities $3996\,bd$, $240\,bd^2$, $4\,b^3d$, and $4\,bd^3$, and inquire, whether the former sum can ever be exactly equal to the latter sum, (in which case $b - d$ would be another root of the equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$,) or whether it must not be always either greater or less than the said latter sum, in which case it will be impossible for a, or $b - d$, or any quantity less than b, or 13, to be a root of the said equation.

Art. 81.

Art. 81. Now, since b is $= 13$, we shall have $b^2 = 169$, and $b^3 = 2197$, and $240\,b^2 d = 40,560\,d$, and $240\,b\,d^2 = 3120\,d^2$, and $3996\,b\,d = 51,948\,d$, and $4\,b^3 d = 8788\,d$, and $6\,b^2 d^2 = 1014\,d^2$, and $4\,b\,d^3 = 52\,d^3$. Therefore the six quantities $14,937\,d + 1998\,d^2 + 240\,b^2 d + 80\,d^3 + 6\,b^2 d^2 + d^4$ will be $(= 14,937\,d + 1998\,d^2 + 40,560\,d + 80\,d^3 + 1014\,d^2 + d^4) = 55,497\,d + 3012\,d^2 + 80\,d^3 + d^4$; and the four quantities $3996\,b\,d + 240\,b\,d^2 + 4\,b^3 d + 4\,b\,d^3$ will be $(= 51,948\,d + 3120\,d^2 + 8788\,d + 52\,d^3) = 60,736\,d + 3120\,d^2 + 52\,d^3$. We must therefore compare the quadrinomial quantity $55,497\,d + 3012\,d^2 + 80\,d^3 + d^4$ with the trinomial quantity $60,736\,d + 3120\,d^2 + 52\,d^3$, and inquire, whether it is possible for the former quantity to be equal to the latter.

Now the quadrinomial quantity $55,497\,d + 3012\,d^2 + 80\,d^3 + d^4$ will be equal to the trinomial quantity $60,736\,d + 3120\,d^2 + 52\,d^3$ if the trinomial quantity $3012\,d^2 + 80\,d^3 + d^4$ is equal (to the quantity $60,736\,d - 55,497\,d + 3120\,d^2 + 52\,d^3$, or) to the trinomial quantity $5239\,d + 3120\,d^2 + 52\,d^3$, or if the binomial quantity $80\,d^3 + d^4$ is equal (to the quantity $5239\,d + 3120\,d^2 - 3012\,d^2 + 52\,d^3$, or) to the trinomial quantity $5239\,d + 108\,d^2 + 52\,d^3$, or if (the quantity $80\,d^3 - 52\,d^3 + d^4$, or) the binomial quantity $28\,d^3 + d^4$ is equal to the binomial quantity $5239\,d + 108\,d^2$, or if the binomial quantity $28\,d\,d + d^3$ is equal to the binomial quantity $5239 + 108\,d$. We must therefore inquire whether it is possible for the binomial quantity

28 $d\,d$

$28\,dd + d^3$ to be equal to the binomial quantity $5239 + 108\,d$.

Art. 82. Now d muſt in this caſe be leſs than b, becauſe it is to be ſubtracted from it. We may therefore ſuppoſe d to increaſe from 0 to b, or 12.756, or 13, but no further.

Now, when d is $= b$, or 13, we ſhall have $108\,d$ $(= 108 \times 13) = 1404$, and dd $(= bb = \overline{13}|^2) = 169$, and d^3 $(= b^3 = \overline{13}|^3) = 2197$, and $28\,dd$ $(= 28 \times 169) = 4732$. Therefore in this caſe the binomial quantity $28\,dd + d^3$ will be $(= 4738 + 2197) = 6929$, and the binomial quantity $5239 + 108\,d$ will be $(= 5239 + 1404) = 6643$; which is leſs than 6929. Therefore, while d increaſes from 0 to 13, or $b - d$ decreaſes from $b - 0$, or b, to $b - b$, or 0, the binomial quantity $28\,dd + d^3$ will increaſe (from $28 \times \overline{0}|^2 + \overline{0}|^3$, or) from 0 to 6929, and the binomial quantity $5239 + 108\,d$ will increaſe from $(5239 + 108 \times 0$, or $5239 + 0$, or) 5239 to $(5239 + 1404$, or) 6643, which is leſs than 6929, or the greateſt value of $28\,dd + d^3$. And conſequently there will be an inſtant of time during the increaſe of the binomial quantity $28\,dd + d^3$ from 0 to 6929 at which, after having been much leſs than the other binomial quantity $5239 + 108\,d$, it will become equal to it: and therefore there will be ſome value of d during it's increaſe from 0 to b, or 13, which will make the quantity $b - d$, or a, be a root of the propoſed equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, or, in other words, the ſaid equation will have a fourth root that will be leſs than b, or 13, or 12.756,441,794,480,744,022. Q. E. I.

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An Investigation of a first near Value of a, *or the fourth, or least, Root of the Equation* $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Art. 83. And, because 6643, or the value of the binomial quantity $5239 + 108d$ when d is $= 13$, is but little less than 6929, or the value of the binomial quantity $28dd + d^3$ at the same time, it is evident that d will be but little less than b, or 13, when the binomial quantity $28dd + d^3$ is exactly equal to the binomial quantity $5239 + 108d$; and consequently we may conclude that $b - d$, or a, will be but little greater than $b - b$, or 0; that is, the fourth root of the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be but little greater than 0, or will be only a very small quantity.

Art. 84. And as a further proof that this fourth, or least, value of x will be only a small quantity and less than 1, we need only suppose x to be $= 1$, and substitute 1 instead of it in the terms of the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$. For, if this be done, we shall find the said compound quantity to be $(= 14,937 \times 1 - 1998 \times 1 \times 1 + 80 \times 1 \times 1 \times 1 - 1 \times 1 \times 1 \times 1 = 14,937 - 1998 + 80 - 1 = 15,017 - 1999) = 13,018$, which is almost triple of 5000, or the absolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, or of the true value of the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ in that equation; and consequently 1 must be much greater than the said true value of x, or the least root of this equation. Q. E. D.

Art. 85. Since x is confiderably lefs than 1, and probably not much greater than $\frac{1}{3}$, or 0.33, the feveral powers of x, to wit, x^2, x^3, and x^4, will be much lefs than x itfelf; and therefore the three terms $1998\,x^2$, $80\,x^3$, and x^4 will be much lefs than $14,937\,x$, or the firft term of this equation; whence we may infer that, in order to obtain a tolerably near value of x, we may fafely fuppofe the firft term alone of the compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ to be equal to the whole four terms, and confequently to the abfolute term 5000, and may deduce the value of x from that fuppofition. And then we fhall have $x = \frac{5000}{14,937} = 0.33$. Therefore 0.33 will be a firft near value of the leaft root of the equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$.

Now, in order to difcover whether 0.33, or this firft near value of x is greater or lefs than it's true value, let it be fubftituted inftead of x in the compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$. And we fhall then have x^2 ($= \overline{0.33}|^2$) $= 0.1089$, and x^3 ($= \overline{0.33}|^3$) $= 0.035,937$, and x^4 ($= \overline{0.33}|^4$) $= 0.011,859,21$, and $14,937\,x$ ($= 14,937 \times 0.33$) $= 4929.21$ and $1998\,x^2$ ($= 1998 \times 0.1089$) $= 217.5822$, and $80\,x^3$ ($= 80 \times 0.035,937$) $= 2.874,960$. Therefore the whole compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ will be ($= 4929.21 - 217.5822 + 2.874,960 - 0.011,859,21 = 4932.084,960 - 217.594,059,21$) $= 4714.490,900,79$, which is confiderably lefs than 5000, or the abfolute term of the propofed equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$. Therefore 0.33 will be lefs

lefs than the true value of x, or the leaft root of that equation. And therefore it feems reafonable to conjecture that the faid true value will be more nearly equal to 0.35; and this fecond near value will be very near the truth, and will be an excellent bafis for a further approach to the true value of x by Mr. Raphfon's method of approximation, which may be made in the manner following:

A further Inveftigation of the Value of the leaft Root of the Biquadratick Equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ *by Mr. Raphfon's Method of Approximation.*

Art. 86. Since 0.35 has been found to be a near value of the leaft root of this equation, let it be fubftituted inftead of x in the terms of the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to difcover whether the value of the faid compound quantity refulting from fuch fubftitution will be greater, or lefs, than 5000, or the abfolute term of the faid equation, and confequently whether 0.35 will be greater, or lefs, than the true value of the leaft root of that equation.

Now, if x is $= 0.35$, we fhall have xx $(= \overline{0.35}|^2) = 0.1225$, and x^3 $(= \overline{0.35}|^3) = 0.042,875$, and x^4 $(= \overline{0.35}|^4) = 0.015,006,25$, and confequently $14,937x$ $(= 14,937 \times 0.35) = 5227.95$, and $1998x^2$ $(= 1998 \times$ $0.1225)$

0.1225) = 244.7550, and $80x^3$ (= 80 × 0.042,875) = 3.430,000, and consequently $14,937x - 1998x^2 + 80x^3 - x^4$ (= 5227.95 − 244.7550 + 3.430,000 − 0.015,006,25 = 5231.380,000 − 244.770,006,25) = 4986.609,993,75; which is somewhat less than 5000, or the absolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore 0.35 is somewhat less than the true value of x, or the least root of the said equation. Q. E. I.

Art. 87. Now let z be put for the unknown difference by which the true value of x exceeds it's near value 0.35. And we shall then have $x = 0.35 + z$, and xx (= $\overline{0.35 + z}|^2 = \overline{0.35}|^2 + 2 \times 0.35 \times z +$ &c) = $0.1225 + 0.70 \times z +$ &c,

and x^3 (= $\overline{0.35 + z}|^3 = \overline{0.35}|^3 + 3 \times \overline{0.35}|^2 \times z +$ &c
$= \overline{0.35}|^3 + 3 \times 0.1225 \times z +$ &c
$= \overline{0.35}|^3 + 0.3675 \times z +$ &c)
$= 0.042,875 + 0.3675 \times z +$ &c,

and x^4 (= $\overline{0.35 + z}|^4 = \overline{0.35}|^4 + 4 \times \overline{0.35}|^3 \times z +$ &c
$= \overline{0.35}|^4 + 4 \times 0.042,875 \times z +$ &c
$= \overline{0.35}|^4 + 0.171,500 \times z +$ &c)
$= 0.015,006,25 + 0.171,500 \times z +$ &c.

Therefore $14,937x$ will be (= $14.937 \times \overline{0.35 + z}$ = $14,937 \times 0.35 + 14,937 \times z$) = $5227.95 + 14,937 \times z$; and $1998xx$ will be (= $1998 \times \overline{0.1225 + 0.70 \times z + \text{\&c}}$ = $1998 \times 0.1225 + 1998 \times 0.70 \times z +$ &c) = 244.7550 + $1,398.60 \times z +$ &c,

and

and $80x^3$ will be $(= 80 \times \overline{0.042,875 + 0.3675 \times z + \&c}$ $= 80 \times 0.042,875 + 80 \times 0.3675 \times z + \&c)$ $= 3.430,000 + 29.4000 \times z + \&c$; and consequently the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be equal to the following compound quantity, to wit,

$$\left\{\begin{array}{llll} & 5227.95 & + & 14,937 \times z \\ - & 244.7550 & - & 1,398.60 \times z - \&c \\ + & 3.430,000 & + & 29.4000 \times z + \&c \\ - & 0.015,006,25 & - & 0.171,500 \times z - \&c, \end{array}\right\}$$

$$= \left\{\begin{array}{lll} 5231.380,000,00 & + & 14,966.400,000 \times z + \&c \\ -244.770,006,25 & - & 1,398.771,500 \times z - \&c \end{array}\right\}$$

$$= 4986.609,993,75 + 13,567.628,500 \times z \ \&c.$$

But the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the binomial quantity $4986.609,993,75 + 13,567.628,500 \times z$ will also be $= 5000$. And consequently $13,567.628,500 \times z$ will be $(= 5000.000,000,00 - 4986.609,993,75) = 13.390,006,25$, and z will be $(= \dfrac{13.390,006,25}{13,567.628,500}) = 0.000,986,9$. Therefore x, or $0.35 + z$, will be $(= 0.35 + 0.000,986,9) = 0.350,986,9$, or, very nearly, $0.350,987$; that is, $0.350,987$ will be a third near value of x, or the least root of the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. Q. E. I.

Note.

Note. All the six figures of this value of x are exact: For, if we were to go through a second process of Mr. Raphson's method of approximation, by supposing x to be equal to 0.350,987 + z, and proceeding as above, we should find z to be = 0.000,000,045,866,14, and consequently the more accurate value of x to be (0.350, 987 + 0.000,000,045,866,14, or) 0.350,987,045,866,14; of which all the figures are probably exact. But I shall content myself on the present occasion with having investigated this least root of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ to the foregoing six figures, 0.350,987.

Of the Two Greatest Roots of the Equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, *of which the Lesser is somewhat less, and the greater is somewhat Greater, than* 32.42.

Art. 88. We will now proceed to investigate to a moderate degree of exactness the two greatest roots of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, which have been shewn above, in art. 79, to be nearly equal to 33.42, the one being a little less, and the other a little greater, than that number. And, first, we will endeavour to find the lesser of the said roots.

Now, since the lesser of these roots is less than 33.42, I will, first, suppose it to be nearly equal to 32, and will

ſubſtitute 32 inſtead of x in the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$, in order to diſcover whether the value of the ſaid compound quantity reſulting from this ſuppoſition will be greater or leſs than 5000, or the abſolute term of the equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$.

Now, if x is $= 32$, we ſhall have $xx\ (= \overline{32}|^2) = 1024$, and $x^3\ (= \overline{32}|^3) = 32{,}768$, and $x^4\ (= \overline{32}|^4) = 1{,}048{,}576$, and $14{,}937\,x\ (= 14{,}937 \times 32) = 477{,}984$, and $1998\,xx\ (= 1998 \times 1024) = 2{,}045{,}952$, and $80\,x^3\ (= 80 \times 32{,}768) = 2{,}621{,}440$, and conſequently $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4\ (= 477{,}984 - 2{,}045{,}952 + 2{,}621{,}440 - 1{,}048{,}576 = 3{,}099{,}424 - 3{,}094{,}528) = 4{,}896$; which is a very little leſs than 5000, or the abſolute term of the propoſed equation. Therefore 32 will be a very little leſs than x.

We will next ſuppoſe x to be $= 32.1$, and ſubſtitute 32.1 inſtead of x in the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$, in order to try whether that value of x will make the value of the ſaid compound quantity approach nearer than the laſt value 4896 to the abſolute term 5000.

Now, if x is $= 32.1$, we ſhall have $xx\ (= \overline{32.1}|^2) = 1030.41$, and $x^3\ (= \overline{32.1}|^3) = 33{,}076.161$, and $x^4\ (= \overline{32.1}|^4) = 1{,}061{,}744.7681$, and $14{,}937\,x\ (= 14{,}937 \times 32.1) = 479{,}477.7$, and $1998\,x^2\ (= 1998 \times 1030.41) = 2{,}058{,}759.18$, and $80\,x^3\ (= 80 \times 33{,}076.161) = 2{,}646{,}092.880$, and conſequently

14,937 x

$14,937x - 1998x^2 + 80x^3 - x^4 = (479,477.7 - 2,058,759.18 + 2,646,092.880 - 1,061,744.7681 = 3,125,570.580 - 3,120,503.9481) = 5066.6319$; which is a little greater than the abſolute term 5000 of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, but differs leſs from it than the former number 4896, ariſing from the ſubſtitution of 32, did. Therefore the true value of x will be greater than 32, but leſs than 32.1, and will probably be nearer to 32.1 than to 32.

Art. 89. We are now in poſſeſſion of three different values of the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, to wit, 4896, 5000, and 5066.6319, which reſult from the ſubſtitution of the number 32, and of the true value of x, and of the number 32.1 in the ſaid compound quantity. We will now therefore, in order to obtain a more accurate value of x than either 32 or 32.1, have recourſe to the doctrine of the Scholium contained above in art. 44, 45, 46, &c - - - 54, and ſuppoſe that the exceſs of the greateſt value of x, to wit, 32.1, above the leaſt value of x, to wit, 32, will be to the exceſs of the true value of x above 32, in nearly the ſame proportion as the exceſs of the greateſt value of the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, (correſponding to 32.1,) to wit, the number 5066.6319, above the leaſt value of the ſaid compound quantity, to wit, the number 4896, (correſponding to 32,) is to the exceſs of the middle value of the ſaid compound quantity, (correſponding to the true value

of

of x,) to wit, the abſolute term 5000 of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4$, above the leaſt value of the ſaid compound quantity, to wit, the number 4896, which correſponds to 32; that is, we will ſuppoſe $32.1 - 32$ to be to $x - 32$ in nearly the ſame proportion as $5066.6319 - 4896$ is to $5000 - 4896$, or 0.1 to be to $x - 32$ in nearly the ſame proportion as 170.6319 is to 104. And then we ſhall have $x - 32 =$, nearly, $\frac{0.1 \times 104}{170.6319}$, or (neglecting the decimal fraction 0.6319 of the denominator 170.6319,) $= \frac{0.1 \times 104}{170} = \frac{10.4}{170} = 0.06$; and conſequently x will be ($= 0.06 + 32$, or) 32.06. Now this will be very nearly equal to the true value of x, and will therefore be an excellent baſis to a further approach to the ſaid true value by means of a proceſs of Mr. Raphſon's method of approximation, which may be performed as follows:

An Inveſtigation of a nearer Value of the Third Root of the Equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ *(which is a little greater than* 32,) *by Mr. Raphſon's Method of Approximation.*

Art. 90. Let the laſt near value of x, to wit, 32.06, be ſubſtituted inſtead of x in the compound quantity $14{,}937x$

$14{,}937x - 1998x^2 + 80x^3 - x^4$, in order to discover whether the value of the said quantity resulting from such substitution will be greater, or less than 5000, or the absolute term of the proposed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, and consequently to determine whether 32.06 is greater, or less, than the true value of x.

Now, if we suppose x to be $= 32.06$, we shall have xx $(= \overline{32.06}|^2) = 1027.8436$, and x^3 $(= \overline{32.06}|^3) = 32{,}952.665{,}816$, and x^4 $(= \overline{32.06}|^4) = 1{,}056{,}462.466{,}060{,}96$, and $14{,}937x$ $(= 14{,}937 \times 32.06) = 478{,}880.22$, and $1998x^2$ $(= 1998 \times 1027.8436) = 2{,}053{,}631.5128$, and $80x^3$ $(= 80 \times 32{,}952.665{,}816) = 2{,}636{,}213.265{,}280$, and consequently $14{,}937x - 1998x^2 + 80x^3 - x^4 = 478{,}880.22 - 2{,}053{,}631.5128 + 2{,}636{,}213.265{,}280 - 1{,}056{,}462.466{,}060{,}96 = 3{,}115{,}093.485{,}280 - 3{,}110{,}093.978{,}860{,}96 = 4999.506{,}419{,}04$; which is a very little less than 5000, or the absolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore 32.06 must be a very little less than the true value of x in that equation.

Q. E. I.

Art. 91. Now let z be put for the unknown difference by which the true value of x, or the third root of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, exceeds it's near value 32.06. And we shall then have $x = 32.06 + z$, and consequently xx $(= \overline{32.06 + z}|^2 = \overline{32.06}|^2 + 2 \times 32.06 \times z + \&c = \overline{32.06}|^2 + 64.12 \times z + \&c) = 1027.8436, + 64.12 \times z + \&c$,

and

and x^3 $(= \overline{32.06 + z}|^3 \times \overline{32.06}|^3 + 3 \times \overline{32.06}|^2 \times z + \&c$

$= \overline{32.06}|^3 + 3 \times 1027.8436 \times z + \&c$

$= \overline{32.06}|^3 + \quad 3083.5308 \times z + \&c)$

$= 32,952.665,816, + 3083.5308 \times z + \&c;$

and x^4 $(= \overline{32.06 + z}|^4 = \overline{32.06}|^4 + 4 \times \overline{32.06}|^3 \times z + \&c$

$= \overline{32.06}|^4 + 4 \times 32,952.665,816 \times z + \&c$

$= \overline{32.06}|^4 + \quad 131,810.663,264 \times z + \&c)$

$= 1,056,462.466,060,96 + 131,810.663,264 + z + \&c.$

Therefore $14,937x$ will be $(= 14,937 \times \overline{32.06 + z}$ $= 14,937 \times 32.06 + 14,937 \times z) = 478,880.22 + 14,937 \times z;$

and $1998x^2$ will be $(= 1998 \times \overline{1027.8436 + 64.12 \times z} + \&c$

$= 1998 \times 1027.8436 + 1998 \times 64.12 \times z + \&c)$

$= 2,053,631.5128 + 128,111.76 \times z + \&c;$

and $80\,x^3$ will be $(= 80 \times$

$\overline{32,952.665,816 + 3083.5308 \times z + \&c} = 80$
$\times\, 32,952.665,816 + 80 \times 3083.5308 \times z + \&c)$
$= 2,636,213.265,280 + 246,682.4640 \times z + \&c;$

and consequently the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be $=$ the compound quantity

$$\left\{ \begin{array}{lll} 478,880.22 & + \; 14,937 \times z & \\ - \; 2,053,631.5128 & - \; 128,111.76 \times z - \&c & \end{array} \right\}$$

$$\left\{ \begin{array}{ll} + \; 2,636,213.265,280 & + \; 246,682.4640 \times z + \&c \\ - \; 1,056,462.466,060,96 & - \; 131,810.663,264 \times z - \&c \end{array} \right\}$$

=

$$=\left\{\begin{array}{l} 3,115,093.485,280 \quad + 261,619.4640 \times z + \&c \\ - 3,110,093.978,860,96 - 259,922.423,264 \times z - \&c \end{array}\right\}$$

= the binomial quantity 4999.506,419,04 + 1,697.040,736 × z &c.

But the compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ is = 5000.

Therefore the binomial quantity 4999.506,419,04 + 1,697.040,736 × z will alſo be = 5000. And conſequently 1697.040,736 × z will be (= 5000.000,000,00 − 4999.506,419,04) = 0.493,580,96, and z will be = $\frac{0.493,580,96}{1697.040,736}$ = 0.000,290,848. Therefore x, or 32.06 + z, will be (= 32.06 + 0.000,290,848) = 32.060,290,848. Q. E. I.

Of this value, 32.060,290,848, of the third root of the propoſed equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$ I am confident that the firſt eight figures 32.060,290, are exact; and the ninth figure 8, is either exact, or too little only by an unit, the ſaid root having been found by another inveſtigation of it to be 32.060, 290,901, &c.

Of the Fourth, or Greatest, Root of the Equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000.$

Art. 92. The fourth, or greatest, root of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ has been shewn above, in art. 79, to be somewhat greater than 33.42. Now, if we suppose it to be as much greater than 33.42 as the third root 32.06 is less than 33.42, it will be = 34.78, or nearly, = 34.8. We will therefore suppose it to be equal to 34.8, and will substitute 34.8 instead of x in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to discover whether the value of that quantity resulting from this substitution will be greater, or less, than 5000, or the absolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and thence to determine whether 32.8 will be greater, or less, than the true value of the greatest root of this equation.

Now, if we suppose x to be equal to 34.8, we shall have xx ($\overline{34.8}|^2$) = 1211.04, and x^3 (= $\overline{34.8}|^3$) = 42,144.192, and x^4 (= $\overline{34.8}|^4$) = 1,466,617.8816. Therefore $14,937x$ will be (= $14,937 \times 34.8$) = 519,807.6, and $1998x^2$ will be (= 1998×1211.04) = 2,419,657.92, and $80x^3$ will be (= $80 \times 42,144.192$) = 3,371,535.360; and consequently the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be

(=

($= 519,807.6 - 2,419,657.92 + 3,371,535.360 - 1,466,617.8816 = 3,891,342.960 - 3,886,275.8016$) $= 5067.1584$; which is somewhat greater than 5000, or the absolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. And hence we may conclude that 34.8 is less (instead of being greater,) than the true value of this greatest root of this equation; because, after x, in it's increase from 0 *ad infinitum*, is become greater than 33.42, the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ decreases to 0, while x increases from 33.42 to the magnitude which it has when the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 0$, or when $1998x^2 + x^4$, after having been less than $14,937x + 80x^3$, becomes equal to it, or $1998x + x^3$ becomes equal to $14,937 + 80x^2$, or $x^3 - 80x^2 + 1998x$ becomes equal to the known quantity 14,937, or while x increases from 33.42 to the quantity that is the root of the cubick equation $x^3 - 80x + 1998x = 14,937$. And, when x is greater than the root of this cubick equation, and increases from thence *ad infinitum*, the binomial quantity $x^4 + 1998x^2$ will always be greater than the binomial quantity $80x^3 + 14,937x$, and the quadrinomial quantity $x^4 + 1998x^2 - 80x^3 - 14,937x$, or $x^4 - 80x^3 + 1998x^2 - 14,937x$, or the excess of $x^4 + 1998x^2$ above $80x^3 + 14,937x$, will increase continually *ad infinitum*. But, not to dwell longer on this nice subject of the alternate increase and decrease of the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ during the increase of x from 0 to a certain magnitude, the decrease of this compound quantity while x increases from 34.8 to some quantity a little greater

An Investigation of a Nearer Value of the Fourth, or Greatest Root of the Equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, *(which is greater than* 34.8,*) by Mr. Raphson's Method of Approximation.*

Art. 95. Let the last near value of x, to wit, 34.832, be substituted instead of x in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to discover whether the value of the said quantity resulting from such substitution will be greater, or less, than 5000, or the absolute term of the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and consequently to determine whether 34.832 will be greater, or less, than the true value of x, or the greatest root of that equation, which we are now investigating.

Now, if we suppose x to be $= 34.832$, we shall have xx $(= \overline{34.832}|^2) = 1213.268,224$, and x^3 $(= \overline{34.832}|^3) = 42,260.558,778,368$, and x^4 $(= \overline{34.832}|^4) = 1,472,019.783,368,114,176$, and $14,937x$ $(= 14,937 \times 34.832) = 520,285.584$, and $1998x^2$ $(= 1998 \times 1213.268,224) = 2,424,109.911,552$, and $80x^3$ $(= 80 \times 42,260.558,778,368) = 3,380,844.702,269,440$. Therefore the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be $(= 520,285.584 - 2,424,109.911,552 + 3,380,844.702,269,440$

269,440 — 1,472,019.783,368,114,176 = 3,901,130.
.286,269,440 — 3,896,129.694,920,114,176) = 5000.591,349,325,824; which is a little greater than 5000, or the abfolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$.

It appears therefore that, while x increafes from 34.8 to 34.832, the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will decreafe from 5067.1584 to 5000.591, 349,325,824, and that, while x increafes further from 34.832 to 34.9, the faid compound quantity will decreafe further from 5000.591,349,325,824 to 4852.8799. And therefore when the faid compound quantity is exactly equal to 5000, the value of x will be greater than 34.832, but lefs than 34.9; that is, the true value of the fourth, or greateft, root of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be fomewhat greater than 34.832.

Art. 96. Now let z be put for the unknown difference by which the true value of x, or the fourth, or greateft, root of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, exceeds it's near value 34.832. And we fhall then have $x = 34.832 + z$, and confequently

$$xx\ (= \overline{34.832 + z}|^2 = \overline{34.832}|^2 + 2 \times 34.832 \times z + \&c$$

$$= \overline{34.832}|^2 + 69.664 \times z + \&c)$$

$$= 1213.268,224 + 69.664 \times z + \&c,$$

and $x^3\ (= \overline{34.832 + z}|^3 = \overline{34.832}|^3 + 3 \times \overline{34.832}|^2 \times z$

$$+ \&c = \overline{34.832}|^3 + 3 \times 1213.268,224 \times z + \&c$$

=

$$= \overline{34.832}|^{3} + 3639.804,672 \times z + \&c)$$
$$= 42,260.558,778,368 + 3639.804,672 \times z + \&c,$$

and x^4 $(= \overline{34.832 + z}|^{4} = \overline{34.832}|^{4} + 4 \times \overline{34.832}|^{3} \times z + \&c = \overline{34.832}|^{4} + 4 \times 42,260.558,778,368 \times z + \&c$
$$= \overline{34.832}|^{4} + 169,042.235,113,472 \times z + \&c)$$
$$= 1,472,019.783,368,114,176 + 169,042.235,113,472 \times z + \&c.$$

Therefore $14,937\,x$ will be $(= 14,937 \times \overline{34.832 + z} = 14,937 \times 34.832 + 14,937 \times z) = 520,285.584 + 14,937 \times z$; and $1998\,x^2$ will be $(= 1998 \times \overline{1213.268,224 + 69.664 \times z + \&c} = 1998 \times 1213.268,224 + 1998 \times 69.664 \times z + \&c) = 2,424,109.911,552 + 139,188.672 \times z + \&c$; and $80\,x^3$ will be $(= 80 \times \overline{42,260.558,778,368 + 3639.804,672 \times z} = 80 \times 42,260.558,778,368 + 80 \times 3639.804,672 \times z + \&c) = 3,380,844.702,269,440 + 291,184.373,760 \times z + \&c$; and confequently the whole compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ will be equal to the compound quantity

$$\left\{\begin{array}{ll} 520,285.584 & + 14,937 \times z \\ -2,424,109.911,552 & - 139,188.672 \times z - \&c \\ +3,380,844.702,269,440 & + 291,184.373,760 \times z + \&c \\ -1,472,019.783,368,114,176 & - 169,042.235,113,472 \times z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{ll} 3,901,130.286,269,440 & + 306,121.373,760 \times z + \&c \\ -3,896,129.694,920,114,176 & - 308,230.907,113,472 \times z - \&c \end{array}\right\}$$

$$= 5000.591.349,325,824 - 2,109.533,353,472 \times z \ \&c.$$

But

But the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the binomial quantity 5000.591,349,325,824 — 2,109.533,353,472 $\times z$ will alfo be = 5000; and confequently 5000.591,349,325,824 will be = 5000 + 2,109.533,353,472 $\times z$, and (fubtracting 5000 from both fides) 2,109.533,353,472 $\times z$ will be = 0.591, 349,325,824, and z will be $= \frac{0.591,349,325,824}{2,109.533,353,472} =$ 0.000,280,32. Therefore x, or 34.832 + z will be (= 34.832 + 0.000,280,32) = 34.832,280,32, or the fourth, or greateft, root of the propofed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be very nearly equal to 34.832,280,32. Q. E. I.

Of this number, 34.832,280,32, I am confident that the firft eight figures, 34.832,280, are exact, having found this root, by another inveftigation of it, to be = 34.832,280,264, &c.

Art. 97. It appears therefore that the four roots of the propofed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ are 0.350,987, &c, and 12.756,441,794,480, 744,022, &c, and 32.060,290,8 &c, and 34.832,280, &c.

Q. E. I.

OF

OF THE DIFFERENCE BETWEEN THE ACCURATE METHODS OF RESOLVING ALGEBRÄICK EQUATIONS AND THE METHODS OF RESOLVING THEM BY APPROXIMATION.

Art. 98. AN *accurate* method of refolving an algebräick equation is a method of refolving it, or inveftigating the value of the unknown quantity contained in it, to any propofed degree of exactnefs by a fingle procefs, or fet of arithmetical operations of addition, fubtraction, multiplication, divifion, or extraction of the fquare-root, carried to a fufficient number of figures to attain the propofed degree of exactnefs. And *a method of refolving an algebräick equation by approximation* is a method of finding a *near* value of the unknown quantity contained in it by a certain procefs of arithmetical operations which, if continued to ever fo great a number of figures, will not approach nearer to the faid value than to a certain limited diftance from it, fo that the further continuation of the faid operations after a certain number of the figures of the faid near value have been obtained, would be intirely ufelefs: and in this circumftance confifts the inferiority of the methods of refolving equations by approximation to the accurate methods of refolving them. Neverthelefs, by repeating the proceffes prefcribed by the methods of approximation we may obtain a fecond and a third and other following near values of the unknown quantity in an equation, that fhall approach continually

continually nearer and nearer to it's true value, till they agree with the said true value to as great a degree, or to as many decimal places of figures, as we please; so that the repetition of the processes in the methods of approximation answers precisely the same purpose as the continuation of the arithmetical operations of the single process in the accurate methods of resolution, by enabling us equally to attain the true value of the unknown quantity to the same degree of exactness. Of this the resolutions of the foregoing three equations by Dr. Halley's and Mr. Raphson's methods of approximation have afforded sufficient proofs. And therefore the methods of resolving equations by approximation are, in point of practical utility, as valuable as the accurate methods of resolving them; and sometimes they are even to be preferred to them, as, for example, in the case of such cubick equations as come within Cardan's two rules, and in that of such biquadratick equations as admit of being resolved by the intervention of cubick equations that come within those rules. For such cubick equations and biquadratick equations may, for the most part, be more easily resolved to any proposed degree of exactness by two, or three, processes of Mr. Raphson's method of approximation than by the application of those rules of Cardan; as has been shewn in some examples of the resolutions of biquadratick and cubick equations in my Appendix to Mr. Frend's Principles of Algebra.

But in the resolution of affected equations, (or equations consisting of more than one term involving the unknown quantity,) that are of a higher order than bi-

quadrätick equations, or that involve the 5th, or 6th, or 7th, or any higher power of x, or the unknown quantity, the methods of approximation are the only methods to which we can resort, there having been no general methods hitherto discovered of resolving such equations accurately, or of obtaining their roots to any proposed degree of exactness by a single process, or course of arithmetical operations. And therefore these methods of resolving equations by approximation must be allowed to be of the greatest and most extensive use in the business of resolving algebräick equations, and the ingenious persons who invented them ought to be reckoned amongst the greatest improvers of this branch of mathematical knowledge. Now the history of the most celebrated of these methods is briefly as follows:

A SHORT ACCOUNT OF THE PRINCIPAL METHODS OF RESOLVING ALGEBRÄICK EQUATIONS BY APPROXIMATION THAT HAVE HITHERTO BEEN PUBLISHED, AND OF THE TIMES OF THEIR PUBLICATION.

99. THE oldest of these methods, as I believe, is that of the great French Algebräist *Vieta*, or Monsieur *Viete*, who flourished about the year 1580, and died in the year 1603 at the age of 63 years, and who is deservedly called *The*

The Father of Algebra, on account of the great variety and importance of his discoveries and improvements in that science. It is contained in a pretty long dissertation, or chapter in his Algebra, intitled, *De numerosâ potestatum purarum atque adfectarum ad Exegesin resolutione Tractatus*, which may be seen in Schooten's edition of his works, published at Leyden in Holland, in the year 1646, of which it takes up 66 pages, from page 163 to page 228, inclusively. This method of resolving equations is a very just and compleat one, and will enable us to find the root of any equation whatsoever to as great a degree of exactness as we please, by repeating the processes sufficiently often for the purpose, and is founded on the same principle as the common method of extracting the square-root of a given number. But it is extreamly laborious in the application of it, on account of the number of substitutions that are to be made in every new process; and it therefore requires a great deal of calculation to determine the root of the proposed equation to any considerable degree of exactness. And for this reason it has seldom been made use of by mathematicians in the resolution of equations for more than a century past, during which time the Publick have been in possession of other methods of doing the same thing with equal certainty and exactness, and with much greater ease and expedition.

Mr. Thomas Harriot, the great English Algebräist, who died in the year 1621, and whose works were published after his death, in the year 1631, and Oughtred, who died at more than 80 years of age in the year 1660,

ſeem to have reſolved high equations by Vieta's method. And ſo did Dr. Pell and Dr. Wallis in his Algebra, (which was publiſhed in the year 1685,) and in all his former works publiſhed in the preceeding 30 years.

100. The next method of reſolving high affected equations by approximation after that of *Vieta*, was, I believe, that of Sir Iſaac Newton, which is briefly deſcribed in his moſt learned little tract, intitled *Analyſis per æquationes numero terminorum infinitas*, which he wrote about the year 1666, when he was only 24 years old, and which he communicated to Dr. Iſaac Barrow, (at that time a Fellow of Trinity-College, Cambridge, and the firſt Lucaſian Profeſſor of Mathematicks in that Univerſity,) in the year 1669. This tract of Sir Iſaac Newton was, in the ſame year 1669, communicated by Dr. Barrow, with the author's permiſſion, to Mr. John Collins; and Mr. Collins, probably, ſhewed it to many of the moſt learned mathematicians of the Royal Society. But it was not publiſhed for above forty years after, when it made part of a very valuable collection of mathematical pieces written by Sir Iſaac Newton, which was publiſhed in a thin quarto volume in the year 1711 by Mr. William Jones, an eminent mathematician in the beginning of the preſent century, and author of the mathematical work intitled, *Synopſis palmariorum Matheſeos*. And it was printed a ſecond time in the following year 1712 in the little octavo volume intitled, *Commercium Epiſtolicum de Analyſi promotâ*, which was printed by the order of the Royal Society in conſequence of the diſpute between Sir Iſaac Newton and Mr. Leibnitz concerning the

the first invention of the Method of Fluxions, or infinitely small Differences. And it is therefore probable that many of the excellent inventions of Sir Isaac Newton contained in that little tract intitled, *Analysis per æquationes numero terminorum infinitas*, remained unknown to the generality of mathematicians until the year 1711. Some of them, however, had been communicated to Dr. John Wallis, Savilian Professor of Geometry in the University of Oxford, several years before, and were mentioned and described by him, in a very concise and summary manner, in his Algebra, which was published in one volume folio at London in the year 1685. And amongst these we find the above-mentioned method of resolving affected equations of all degrees by approximation. See Wallis's Algebra, pages 338, 339.

101. The next method of approximating to the roots of affected equations that was made publick to the world was that of Mr. Joseph Raphson. This method was published in the year 1690, under the title of *Analysis Æquationum Universalis, seu ad Æquationes Algebräicas resolvendas Methodus generalis et expedita*, in a thin quarto volume containing about 50 pages. It is a very excellent method of resolving all sorts of equations by approximation in a much more easy and expeditious manner than that of *Vieta*, and is illustrated by a great number of curious and judicious examples. But it differs so little from that of Sir Isaac Newton, of which Dr. Wallis had published a specimen in his Algebra in the year 1685, that one can hardly avoid conjecturing that it was suggested to it's ingenious inventor by the perusal of

that ſpecimen. But, from whatever ſource it might take it's riſe, it was a very valuable treatiſe, and was ſo fully illuſtrated by examples that it appears to have been very generally read and adopted by mathematicians, and deſervedly conſidered by them as a great improvement in Algebra. And accordingly it ſoon after became the baſis, or ground-work, of two further improvements, or ſuppoſed improvements, in this branch of Algebra, the one in France, and the other in England, to wit, the method of reſolving equations by approximation publiſhed in the Memoirs of the French Academy of Sciences in a year or two after, by *Monſieur de Lagny*, a Profeſſor of Mathematicks in the Univerſity of Paris, and that of the celebrated Mr. Edmund Halley, (afterwards Dr. Halley,) publiſhed at London in the Philoſophical Tranſactions for the year 1694, and afterwards in the year 1708 in the ſecond volume of the *Miſcellanea Curioſa*, and which is now publiſhed again at the beginning of this diſcourſe. For both theſe methods of approximating to the roots of equations ſeem manifeſtly to have been derived from Mr. Raphſon's method; and Dr. Halley confeſſes his to have been ſo. And, indeed, they are little more than variations of Mr. Raphſon's method, and ariſe ſo obviouſly from it, that one would naturally ſuppoſe that they muſt have occurred to Mr. Raphſon himſelf after he had diſcovered and perfectly underſtood his own method. And therefore it ſeems probable that, after he had carefully examined theſe methods, he did not think them, upon the whole, ſo convenient and fit to be adopted in practice as his own method. And this, (as he informs us in a ſubſequent edition of his book publiſhed in the year 1697,

after

after the publication of this tract of Dr. Halley,) was really the case; he being of opinion, "that the greater "simplicity of his own method, which proceeds by the "resolution of only simple equations, made it preferable "to the methods of *Monsieur de Lagny* and Dr. Halley, "which proceed either by the admission of the square of "the unknown quantity of the second, or transformed, "equation into the expression that denotes the near value "of it, after having first found the said expression for the "near value by the resolution of a simple equation, or by "resolving a quadratick equation at first instead of a "simple equation." And in this opinion, after a good deal of attention to the subject, and many trials of these different methods of approximating to the roots of equations, I am much inclined to agree with him; and I presume that, after the perusal of the resolutions of the three equations above-mentioned both by Dr. Halley's and by Mr. Raphson's methods of approximation, which have been given at great length in this Appendix, a majority of the readers of this tract will also be of the same opinion.

THE END.

DR. WALLIS's SOLUTION

OF

COLONEL TITUS's ARITHMETICAL PROBLEM.

An ARITHMETICAL PROBLEM, *proposed to* Dr. WALLIS *by* Colonel SILAS TITUS.

Article 1. IN the foregoing tract of Dr. Halley on the resolution of high affected equations by approximation, the third example adduced by him to illustrate his method of approximation, is the biquadratick equation $14,937z - 1998z^2 + 80z^3 - z^4 = 5000$, or $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$; which, he informs us, was obtained by Dr. Wallis in the 62nd chapter of his Algebra as the result of his solution of a very difficult arithmetical problem, and which Dr. Wallis has there resolved, to a great degree of exactness, by the method of resolving numeral equations, taught by Vieta in pages 163, 164, &c to page 228, of Schooten's edition of his mathematical works, published in the year 1646. See above, page 17. Now, as this equation has been so much the object of our attention both in Dr. Halley's own tract above-mentioned on the resolution of equations by approximation, and in my appendix to it, (in which latter tract I have investigated all it's four roots, 0.350,987,046, &c, 12.756,441,794,480,744,02 &c, 32.060,290,8 &c, and 34 832,280,2 &c,) I presume that my readers will

not

not be ſorry to ſee Dr. Wallis's Solution of the Problem that gave riſe to it; more eſpecially as I ſhall endeavour to ſet forth every ſtep in the ſolution (which is very long and intricate,) in the fulleſt and cleareſt manner poſſible, ſo that the ſolution will be much more intelligible, as it will be here exhibited, than as it is exhibited in Dr. Wallis's own book, and my readers will therefore be ſpared a great deal of time and pains that I have found it neceſſary to beſtow on it before I could thoroughly underſtand it.

Art. 2. This problem was propoſed to Dr. Wallis in the year 1662, by Colonel Silas Titus, a gentleman of the bed-chamber to King Charles the Second, who was diſtinguiſhed for his knowledge of the Mathematicks, as well as of other branches of uſeful learning; and it had been originally propoſed to Colonel Titus (as the Colonel informed Dr. Wallis,) by Dr. John Pell, the famous Algebräiſt of that time. It was as follows:

AN ARITHMETICAL PROBLEM.

There are three numbers of ſuch magnitudes that the ſquare of the firſt number together with the product of the multiplication of the ſecond and third numbers into each other is equal to the number 16; and that the ſquare of the ſecond number together with the product of the multiplication of the firſt and third numbers into each other

other is equal to the number 17; and that the ſquare of the third number together with the product of the multiplication of the firſt and ſecond numbers into each other is equal to the number 18. It is required to find the ſaid three numbers.

THE SOLUTION.

BY DR. JOHN WALLIS,

SAVILIAN PROFESSOR OF GEOMETRY AT OXFORD, IN JUNE, 1662.

Art. 3. Let a be put for the firſt of theſe numbers, b for the ſecond, and c for the third. Then will the equation $aa + bc = 16$ expreſs the firſt condition of the problem; and the equation $bb + ac = 17$ will expreſs the ſecond condition of it; and the equation $cc + ab = 18$ will expreſs it's third condition: ſo that each of the three equations that expreſs the three conditions of the problem will involve in it all the three unknown quantities a, b, and c. And hence ariſes the great difficulty of ſolving the problem; it being neceſſary to perform a great number of laborious Algebräical operations, before we can diſentangle theſe three unknown quantities from each other, and derive from the three aforeſaid original equations $aa + bc = 16$, $bb + ac = 17$, and $cc + ab = 18$ (each of which involves all the three unknown quantities,) an equation that ſhall involve only one of them.

Art. 4.

Art. 4. The firſt thing to be ſhewn concerning theſe three unknown numbers is, " that each of them is different from the other two," or " that no two of them are equal to each other." This Dr. Wallis proves in the following manner.

In the firſt place, the firſt number a cannot be equal to the ſecond number b. For, if it were ſo, aa would be equal to bb, and bc would be $= ac$, and conſequently $aa + bc$ would be $= bb + ac$. But, by the 1ſt condition of the problem, $aa + bc$ is $= 16$; and, by the 2nd condition of the problem, $bb + ac$ is $= 17$. Therefore 16 would be $= 17$; which is impoſſible. Therefore the ſuppoſition " that a is equal to b," from which this impoſſible concluſion is deduced, cannot be true. Q. E. D.

Secondly, the firſt number a cannot be equal to the third number c. For, if it were ſo, aa would be $= cc$, and bc would be $= ba$, and conſequently $= ab$, and conſequently $aa + bc$ would be $= cc + ab$. But, by the 1ſt condition of the problem, $aa + bc$ is $= 16$; and, by the 3d condition of the problem, $cc + ab$ is $= 18$. Therefore 16 would be $= 18$; which is impoſſible. Therefore the ſuppoſition " that a is equal to c," from which this impoſſible concluſion is deduced, cannot be true. Q. E. D.

Thirdly, the ſecond number b cannot be equal to the third number c. For, if it were ſo, bb would be equal to cc, and ac would be equal to ab, and conſequently $bb + ac$ would be $= cc + ab$. But, by the ſecond condition

dition of the problem, $bb + ac$ is $= 17$; and, by the third condition of it, $cc + ab$ is $= 18$. Therefore 17 would be $= 18$; which is impoſſible. Therefore the ſuppoſition "that b is equal to c," from which this impoſſible concluſion is deduced, cannot be true. Q. E. D.

Art. 5. Having thus ſhewn that the three unknown numbers a, b, and c muſt be all different one from the other, it will be expedient, in the next place, to ſhew that c, or the third number, of which the ſquare cc occurs in the third equation $cc + ab = 18$, will be greater than either of the other two numbers a and b. This may be proved in the manner following:

Since the three unknown numbers a, b, and c are all different from each other, let the leaſt of them be denoted by the ſmall Greek letter α; and let the exceſs of the middle number above the leaſt be denoted by the ſmall Greek letter β, and the exceſs of the greateſt number above the middle one be denoted by the ſmall Greek letter γ. Then will the ſecond, or middle, number be $= \alpha + \beta$, and the third, or greateſt, number will be $= \alpha + \beta + \gamma$.

Therefore the product of the multiplication of the middle number into the greateſt number will be $= \overline{\alpha + \beta|} \times \overline{\alpha + \beta + \gamma} = \alpha^2 + \alpha\beta + \alpha\gamma + \beta\alpha + \beta^2 + \beta\gamma) = \alpha^2 + 2\alpha\beta + \alpha\gamma + \beta\gamma + \beta^2$, and the ſum of the ſquare of the leaſt number and the ſaid product of the other two numbers will be $(\alpha^2 + \alpha^2 + 2\alpha\beta + \alpha\gamma + \beta\gamma + \beta^2$, or$)$ $2\alpha^2 + 2\alpha\beta + \alpha\gamma + \beta\gamma + \beta^2$.

The

The ſquare of the middle number $\alpha + \beta$ will be $\alpha^2 + 2\alpha\beta + \beta^2$; and the product of the multiplication of the other two numbers α and $\alpha + \beta + \gamma$ into each other will be $\alpha^2 + \alpha\beta + \alpha\gamma$. Therefore the ſum of the ſaid ſquare and product will be $2\alpha^2 + 3\alpha\beta + \beta^2 + \alpha\gamma$.

The ſquare of the greateſt number $\alpha + \beta + \gamma$ will be $(\alpha^2 + \alpha\beta + \alpha\gamma + \beta\alpha + \beta^2 + \beta\gamma + \gamma\alpha + \gamma\beta + \gamma^2$, or) $\alpha^2 + 2\alpha\beta + 2\alpha\gamma + \beta^2 + 2\beta\gamma + \gamma^2$; and the product of the multiplication of the other two numbers α and $\alpha + \beta$ into each other will be $\alpha^2 + \alpha\beta$. Therefore the ſum of the ſaid ſquare and product will be $= 2\alpha^2 + 3\alpha\beta + 2\alpha\gamma + \beta^2 + 2\beta\gamma + \gamma^2$.

And this laſt ſum is greater than either of the two former ſums, $2\alpha^2 + 2\alpha\beta + \alpha\gamma + \beta\gamma + \beta^2$ and $2\alpha^2 + 3\alpha\beta + \beta^2 + \alpha\gamma$; it's exceſs above the firſt of theſe ſums, to wit, $2\alpha^2 + 2\alpha\beta + \alpha\gamma + \beta\gamma + \beta^2$, being $\alpha\beta + \alpha\gamma + \beta\gamma + \gamma^2$, and it's exceſs above the ſecond ſum, to wit, $2\alpha^2 + 3\alpha\beta + \beta^2 + \alpha\gamma$, being $\alpha\gamma + 2\beta\gamma + \gamma^2$.

Now the leaſt of theſe three ſums muſt, by the conditions of the problem, be equal to 16, and the leaſt but one muſt be equal to 17, and the greateſt muſt be equal to 18.

Therefore, ſince the laſt ſum $2\alpha^2 + 3\alpha\beta + 2\alpha\gamma + \beta^2 + 2\beta\gamma + \gamma^2$, (which is $= \overline{\alpha + \beta + \gamma}|^2 + \alpha \times \overline{\alpha + \beta}$,) is greater than either of the other two, it muſt be $= 18$; that is, $\overline{\alpha + \beta + \gamma}|^2 + \alpha \times \overline{\alpha + \beta}$ muſt be $= 18$.

But

But $cc + ab$ is $= 18$.

Therefore $\overline{\alpha + \beta + \gamma}^2 + \alpha \times \overline{\alpha + \beta}$ muſt be $= cc + ab$; and conſequently c muſt be $= \alpha + \beta + \gamma$, or muſt be the greateſt of the three unknown numbers a, b, and c. Q. E. D.

Art. 6. But, whether the firſt unknown number a (of which the ſquare occurs in the firſt equation $aa + bc = 16$,) or the ſecond unknown number b, (of which the ſquare occurs in the ſecond equation $bb + ac = 17$,) is the greater, or is equal to the binomial quantity $\alpha + \beta$, does not yet appear. But Dr. Wallis proves that b is the greater, or is equal to $\alpha + \beta$, and conſequently that a is equal to α, or the leaſt of the three unknown quantities a, b, and c, by the following train of reaſoning.

Since the ſextinomial quantity $2\alpha^2 + 3\alpha\beta + 2\alpha\gamma + \beta^2 + 2\beta\gamma + \gamma^2$ (which is $= \overline{\alpha + \beta + \gamma}|^2 + \alpha \times \overline{\alpha + \beta}$) is $= 18$, and either the quinquinomial quantity $2\alpha^2 + 2\alpha\beta + \alpha\gamma + \beta\gamma + \beta^2$ (which is $= \alpha^2 + \overline{\alpha + \beta}| \times \overline{\alpha + \beta + \gamma}$), or the quadrinomial quantity $2\alpha^2 + 3\alpha\beta + \beta^2 + \alpha\gamma$, (which is $= \overline{\alpha + \beta}|^2 + \alpha \times \overline{\alpha + \beta + \gamma}$,) is $= 17$, and the other of the ſaid two quantities is $= 16$; and the numbers 18, 17, and 16 form an arithmetical progreſſion, (the exceſs of 18 above 17, being equal to the exceſs of 17 above 16, to wit, 1);—it follows that the three quantities $2\alpha^2 + 3\alpha\beta + 2\alpha\gamma + \beta^2 + 2\beta\gamma + \gamma^2$,

and

and $2\alpha^2 + 2\alpha\beta + \alpha\gamma + \beta\gamma + \beta^2$ and $2\alpha^2 + 3\alpha\beta + \beta^2 + \alpha\gamma$, (of which the firſt is equal to 18, and either the ſecond or the third is equal to 17, and either the third or the ſecond is equal to 16,) muſt alſo, if the ſecond and third were to be placed in their proper order, form an arithmetical progreſſion. Therefore, if from theſe three quantities we ſubtract the quantity $2\alpha^2 + 2\alpha\beta + \alpha\gamma + \beta^2$, which is common to them all, the remainders after ſuch ſubtractions muſt alſo, when the ſecond and third remainders are placed in their proper order, form an arithmetical progreſſion; that is, the quadrinomial quantity, $\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma^2$, the ſingle quantity $\beta\gamma$, and the ſingle quantity $\alpha\beta$, will, when $\beta\gamma$ and $\alpha\beta$ are placed in their proper order, form an arithmetical progreſſion. It remains only to be determined, whether $\beta\gamma$ is to be placed immediately after the quadrinomial quantity $\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma^2$, and before $\alpha\beta$, or is to be placed third in order and after $\alpha\beta$. Now, if $\beta\gamma$ was to be placed ſecond, or immediately after $\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma^2$, and before $\alpha\beta$, it would follow from the nature of an arithmetical progreſſion (in which twice the middle term, when the number of terms is odd, is always equal to the ſum of the two terms adjoining to the middle term), that $2\beta\gamma$ would be equal to the ſum of $\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma^2$ and $\alpha\beta$, that is, to $2\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma^2$. But this is impoſſible, becauſe $2\beta\gamma$ makes only a part of that quantity. Therefore $\beta\gamma$ muſt not be placed next to $\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma^2$, but muſt be third in order, and muſt come after $\alpha\beta$; and conſequently the three terms of this arithmetical progreſſion muſt be firſt, $\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma^2$, and, ſecondly, $\alpha\beta$, and thirdly, $\beta\gamma$. Therefore the former arithmetical

arithmetical progreſſion, (from which this progreſſion was derived by ſubtracting the quadrinomial quantity $2\alpha^2 + 2\alpha\beta + \beta^2 + \alpha\gamma$ from each of it's terms,) will conſiſt of the three following terms, to wit, firſt, $2\alpha^2 + 3\alpha\beta + 2\alpha\gamma + 2\beta\gamma + \beta^2 + \gamma^2$, and, ſecondly, $2\alpha^2 + 3\alpha\beta + \beta^2 + \alpha\gamma$, and thirdly, $2\alpha^2 + 2\alpha\beta + \beta^2 + \alpha\gamma + \beta\gamma$. Therefore $2\alpha^2 + 3\alpha\beta + \beta^2 + \alpha\gamma$ will be $= 17$, and $2\alpha^2 + 2\alpha\beta + \beta^2 + \alpha\gamma + \beta\gamma$ will be $= 16$; that is, $\overline{\alpha + \beta}|^2 + \alpha \times \overline{\alpha + \beta + \gamma}$ will be $= 17$, and $\alpha^2 + \overline{\alpha + \beta} \times \overline{\alpha + \beta + \gamma}$ will be $= 16$. Therefore $\overline{\alpha + \beta}|^2 + \alpha \times \overline{\alpha + \beta + \gamma}$, or $\overline{\alpha + \beta}|^2 + \alpha \times c$, will be $(= 17) = bb + ac$; and conſequently b will be $= \alpha + \beta$, or will be the ſecond of the three quantities α, $\alpha + \beta$, and $\alpha + \beta + \gamma$, or a, b, and c; and $\alpha^2 + \overline{\alpha + \beta} \times \overline{\alpha + \beta + \gamma}$, or $\alpha^2 + b \times c$, or $\alpha^2 + bc$, will be $(= 16) = aa + bc$, and conſequently a will be $= \alpha$, or will be the leaſt of the three unknown quantities α, $\alpha + \beta$, and $\alpha + \beta + \gamma$, or a, b, and c. Q. E. D.

We may therefore now conclude that a will be leſs than b, as well as that b will be leſs than c; which concluſion will be found uſeful in the courſe of the following inveſtigation of the value of a, or the firſt of the three unknown quantities a, b, and c, which is that which Dr. Wallis makes the object of his purſuit, and by means of which he afterwards determines the two other quantities b and c, and gives a compleat ſolution of the problem.

Art. 7. Now, for the ſake of brevity, let the letter l be ſubſtituted inſtead of 16 in the equation $aa + bc = 16$, and

and the letter m be ſubſtituted inſtead of 17 in the equation $bb + ac = 17$, and the letter n be ſubſtituted inſtead of 18 in the equation $cc + ab = 18$. And then thoſe three original equations will be $aa + bc = l$, $bb + ac = m$, and $cc + ab = n$.

Now, ſince $aa + bc$ is $= l$, we ſhall have $bc = l - aa$, and conſequently $c = \frac{l - aa}{b}$, and cc $(= \overline{\frac{l - aa}{bb}}|^2) = \frac{l^2 - 2la^2 + a^4}{bb}$.

Secondly, ſince $cc + ab$ is $= n$, we ſhall have $cc = n - ab$.

Therefore, 3dly, $n - ab$ will be $= \frac{l^2 - 2la^2 + a^4}{bb}$, and conſequently $nbb - ab^3$ will be $= l^2 - 2la^2 + a^4$.

Therefore (adding ab^3 to both ſides) nbb will be $= l^2 - 2la^2 + a^4 + ab^3$, and (adding $2la^2$ to both ſides,) $nbb + 2la^2$ will be $= l^2 + a^4 + ab^3$, and (ſubtracting $l^2 + a^4$ from both ſides,) ab^3 will be $= nbb - l^2 + 2la^2 - a^4$, and conſequently bb will be $= \frac{nbb - l^2 + 2la^2 - a^4}{ab}$.

But $bb + ac$ is $= m$, and conſequently bb is $= m - ac$.

Therefore

Therefore $\frac{nbb - l^2 + 2la^2 - a^4}{ab}$ (which is $= bb$,) will alſo be $= m - ac$.

But it has been ſhewn that c is $= \frac{l - aa}{b}$.

Therefore ac will be $(= a \times \frac{l-aa}{b}) = \frac{la - a^3}{b}$, and $m - ac$ will be $= m - \frac{la - a^3}{b}$ $(= \frac{mb - \overline{la - a^3}}{b} = \frac{mb - la + a^3}{b}) = \frac{mab - la^2 + a^4}{ab}$.

Therefore $\frac{nbb - l^2 + 2la^2 - a^4}{ab}$, (which has been ſhewn to be $m - ac$,) will be $= \frac{mab - la^2 + a^4}{ab}$; and conſequently (multiplying both ſides into ab) $nbb - l^2 + 2la^2 - a^4$ will be $= mab - la^2 + a^4$.

Therefore (adding $l^2 + a^4$ to both ſides,) we ſhall have $nbb + 2la^2 = mab - la^2 + 2a^4 + ll$, and (ſubtracting $2la^2$ from both ſides,) $nbb = mab - 3la^2 + 2a^4 + l^2$.

Now, ſince a has been ſhewn in art. 6 to be leſs than b, and m is leſs than n, (m being $= 17$ and $n = 18$) it follows that mab will be leſs than nbb, and conſequently than the quadrinomial quantity $mab - 3la^2 + 2a^4 + l^2$, which is equal to nbb. Therefore mab may be ſubtracted

from

from both ſides of the laſt equation $nbb = mab - 3la^2 + 2a^4 + l^2$. Let it be ſo ſubtracted. And we ſhall then have $nbb - mab = 2a^4 - 3la^2 + l^2$. Therefore (dividing both ſides by n,) we ſhall have $bb - \frac{ma}{n} \times b = \frac{2a^4 - 3la^2 + l^2}{n}$.

Art. 8. Now let $\frac{m^2a^2}{4n}$, or the ſquare of $\frac{ma}{2n}$, be added to both ſides of the laſt equation. And we ſhall then have $bb - \frac{ma}{n} \times b + \frac{m^2a^2}{4n^2} = \frac{2a^4 - 3la^2 + l^2}{n} + \frac{m^2a^2}{4n^2} = \frac{4n \times \overline{2a^4 - 3la^2 + l^2}}{4n \times n} + \frac{m^2a^2}{4n^2} = \frac{8na^4 - 12lna^2 + 4l^2n}{4n^2} + \frac{m^2a^2}{4n^2} = \frac{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}{4n^2}$. Therefore (extracting the ſquare-roots of both ſides,) we ſhall have $b - \frac{ma}{2n} = \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$, and conſequently $b = \frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$.

Therefore (ſquaring both ſides of this equation,) we ſhall have $bb = \frac{m^2a^2}{4n^2} + \frac{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}{4n^2}$

+

$$+ \frac{2ma}{2n} \times \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n} =$$

$$\frac{8na^4 - 12lna^2 + 2m^2a^2 + 4l^2n}{4n^2} + \frac{2ma}{2n} \times$$

$$\frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n} =$$

$$\frac{8na^4 - 12lna^2 + 2m^2a^2 + 4l^2n}{4n^2} +$$

$$\frac{2ma \times \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{4n^2}.$$

But $bb + ac$ is $= m$. And ac has been shewn to be $= \frac{la - a^3}{b}$.

Therefore $bb + \frac{la - a^3}{b}$ will be $= m$.

Therefore the foregoing value of bb together with $\frac{la - a^3}{b}$ will be $= m$.

But b has been shewn to be $=$

$$\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n} =$$

$$\frac{ma + \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}.$$

Therefore $\frac{la - a^3}{b}$ will be $= \overline{la - a^3} \times$ the reciprocal of this value of b, that is, to $\overline{la - a^3} \times$ the

fraction

fraction $\dfrac{2n}{ma + \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}$, or to the fraction $\dfrac{2lna - 2na^3}{ma + \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}$.

Therefore the foregoing value of bb together with this last fraction will be $= m$; that is, the quantity

$$\frac{8na^4 - 12lna^2 + 2m^2a^2 + 4l^2n}{4n^2} +$$

$$\frac{2ma \times \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{4n^2} +$$

$$\frac{2lna - 2na^3}{ma + \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}} \text{ will be } = m.$$

Art. 9. To abridge this long expression, let E^2 be put $= 8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, and consequently $E = \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}$. And then we shall have $8na^4 - 12ln + 2m^2a^2 + 4l^2n$ ($= 8na^4 - 12ln + m^2a^2 + 4l^2n + m^2a^2$) $= E^2 + m^2a^2$; and consequently the last equation will become $\dfrac{E^2 + m^2a^2}{4n^2} + \dfrac{2ma \times E}{4n^2} + \dfrac{2lna - 2na^3}{ma + E} = m$. Therefore (if we multiply both sides of the equation by $4n^2$,) we shall have $E^2 + m^2a^2 + 2maE + \dfrac{8ln^3a - 8n^3a^3}{ma + E} = 4mn^2$; and (multiplying both sides by $ma + E$) we shall have $maE^2 + m^3a^3 + 2m^2a^2E + E^3 + m^2a^2E + 2maE^2 + 8ln^3a - 8n^3a^3 = 4m^2n^2a + 4mn^2E$; and (subtracting $4m^2n^2a$ from both sides) $maE^2 + m^3a^3 + 2m^2a^2E + E^3 + m^2a^2E$

+

$+ 2ma\text{E}^2 + 8ln^3a - 8n^3a^3 - 4m^2n^2a = 4mn^2\text{E}$, or $\text{E}^3 + 3ma\text{E}^2 + 3m^2a^2\text{E} + m^3a^3 + 8ln^3a - 8n^3a^3 - 4m^2n^2a = 4mn^2\text{E}$; and (subtracting $\text{E}^3 + 3m^2a^2\text{E}$ from both sides) $3ma\text{E}^2 + m^3a^3 + 8ln^3a - 8n^3a^3 - 4m^2n^2a = 4mn^2\text{E} - 3m^2a^2\text{E} - \text{E}^3$, and (substituting, instead of E^2 in the term $3ma\text{E}^2$, it's value $8na^4 - 12lna^2 + m^2a^2 + 4l^2n$) $24mna^5 - 36lmna^3 + 3m^3a^3 + 12l^2mna + m^3a^3 + 8ln^3a - 8n^3a^3 - 4m^2n^2a = 4mn^2\text{E} - 3m^2a^2\text{E} - \text{E}^3$, or $24mna^5 - 36lmna^3 + 4m^3a^3 - 8n^3a^3 + 12l^2mna + 8ln^3a - 4m^2n^2a = 4mn^2\text{E} - 3m^2a^2\text{E} - \text{E}^3$. Therefore the square of the septinomial quantity $24mna^5 - 36lmna^3 + 4m^3a^3 - 8n^3a^3 + 12l^2mna + 8ln^3a - 4m^2n^2a$ will be equal to the square of the trinomial quantity $4mn^2\text{E} - 3m^2a^2\text{E} - \text{E}^3$; that is, the compound quantity

$$\left\{\begin{array}{lll}
576m^2n^2a^{10} - 1728lm^2n^2a^8 & + 1872l^2m^2n^2a^6 & \\
\quad + 192m^4na^8 & + 960lmn^4a^6 & \\
\quad - 384mn^4a^8 & - 256m^3n^3a^6 & \\
 & - 288lm^4na^6 & \\
 & + 16m^6a^6 & \\
 & + 64n^6a^6 & \\
- 864l^3m^2n^2a^4 & + 144l^4m^2n^2a^2 & \\
- 768l^2mn^4a^4 & + 192l^3mn^4a^2 & \\
+ 352lm^3n^3a^4 & - 96l^2m^3n^3a^2 & \\
+ 96l^2m^4na^4 & + 64l^2n^6a^2 & \\
- 32m^5n^2a^4 & - 64lm^2n^5a^2 & \\
- 128ln^6a^4 & + 16m^4n^4a^2 & \\
+ 64m^2n^5a^4 & &
\end{array}\right\}$$

will be equal to the compound quantity

$9m^4$

$$\left\{\begin{array}{lll}
 & +\ 72m^4na^8 & -\ 108lm^4na^6 \\
 & & +\ 9m^6a^6 \\
 & & -\ 192m^3n^3a^6 \\
384m^2n^2a^{10} & -\ 1152lm^2n^2a^8 & +\ 1248l^2m^2n^2a^6 \\
 & +\ 96m^4na^8 & -\ 144lm^4na^6 \\
 & -\ 512mn^4a^8 & +\ 6m^6a^6 \\
 & & +\ 1536lmn^4a^6 \\
 & & -\ 128m^3n^3a^6 \\
+\ 36l^2m^4na^4 & -\ 96l^2m^3n^3a^2 & \\
+\ 288lm^3n^3a^4 & -\ 192lm^2n^5a^2 & \\
-\ 24m^5n^2a^4 & +\ 16m^4n^4a^2 & +\ 64l^2m^2n^5 \\
+\ 128m^2n^5a^4 & +\ 96l^4m^2n^2a^2 & \\
-\ 576l^3m^2n^2a^4 & +\ 768l^3mn^4a^2 & \\
+\ 48l^2m^4na^4 & -\ 64l^2m^3n^3a^2 & -\ 128l^4mn^4 + E^6. \\
-\ 1664l^2mn^4a^4 & & \\
+\ 192lm^3n^3a^4 & & \\
-\ 8m^5n^2a^4 & &
\end{array}\right\}$$

or to the compound quantity

$$\left\{\begin{array}{lll}
384m^2n^2a^{10} & +\ 168m4na^8 & -\ 252lm^4na^6 \\
 & -\ 1152lm^2n^2a^8 & +\ 15m^6a^6 \\
 & -\ 512mn^4a^8 & -\ 320m^3n^3a^6 \\
 & & +\ 1248l^2m^2n^2a^6 \\
 & & +\ 1536lmn^4a^6 \\
+\ 84l^2m^4na^4 & -\ 160l^2m^3n^3a^2 & \\
+\ 480lm^3n^3a^4 & -\ 192lm^2n^5a^2 & \\
-\ 32m^5n^2a^4 & +\ 16m^4n^4a^2 & \\
+\ 128m^2n^5a^4 & +\ 96l^4m^2n^2a^2 & \\
-\ 576l^3m^2n^2a^4 & +\ 768l^3mn^4a^2 & \\
-\ 1664l^2mn^4a^4 & & +\ 64l^2m^2n^5 \\
 & & -\ 128l^4mn^4 + E^6.
\end{array}\right\}$$

Lastly,

Laftly, fince E² is $= 8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, and E⁴ is = the compound quantity

$$\left\{ \begin{array}{cccccccc} 64n^2a^8 & - & 192ln^2a^6 & + & 208l^2n^2a^4 & - & 96l^3n^2a^2 & \\ & + & 16m^2na^6 & - & 24lm^2na^4 & + & 8l^2m^2na^2 & \\ & & & + & m^4a^4 & & + 16l^4n^2, & \end{array} \right\}$$

we fhall have $E^6 =$ to the product of the multiplication of this compound quantity into the quadrinomial quantity $8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, that is, to the compound quantity

$$\left\{ \begin{array}{llllll} 512n^3a^{12} & - & 2304ln^3a^{10} & + & 4224l^2n^3a^8 & \\ & + & 192m^2n^2a^{10} & - & 576lm^2n^2a^8 & \\ & & & + & 24m^4na^8 & \\ - 4032l^3n^3a^6 & + & 2112l^4n^3a^4 & - & 576l^5n^3a^2 & \\ + 624l^2m^2n^2a^6 & - & 288l^3m^2n^2a^4 & + & 48l^4m^2n^2a^2 & \\ - 36lm^4na^6 & + & 12l^2m^4na^4 & & & \\ + m^6a^6 & & & + & 64l^6n^3. & \end{array} \right\}$$

Therefore the compound quantity

$$\left\{ \begin{array}{lll} 9m^4a^4E^2 & + 6m^2a^2E^4 & \\ - 24m^3n^2a^2E^2 & - 8mn^2E^4 & + E^6 \\ + 16m^2n^4E^2 & & \end{array} \right\}$$

will be equal to the following very numerous compound quantity, to wit,

$$
\left\{
\begin{array}{lll}
 & +\ 72m^4na^8 & -\ 108lm^4na^6 \\
 & & +\ 9m^6a^6 \\
 & & -\ 192m^3n^3a^6 \\
384m^2n^2a^{10} & -\ 1152lm^2n^2a^8 & +\ 1248l^2m^2n^2a^6 \\
 & +\ 96m^4na^8 & -\ 144lm^4na^6 \\
 & -\ 512mn^4a^8 & +\ 6m^6a^6 \\
 & & +\ 1536lmn^4a^6 \\
 & & -\ 128m^3n^3a^6 \\
+\ 36l^2m^4na^4 & -\ 96l^2m^3n^3a^2 & \\
+\ 288lm^3n^3a^4 & -\ 192lm^2n^5a^2 & \\
-\ 24m^5n^2a^4 & +\ 16m^4n^4a^2 & +\ 64l^2m^2n^5 \\
+\ 128m^2n^5a^4 & +\ 96l^4m^2n^2a^2 & \\
-\ 576l^3m^2n^2a^4 & +\ 768l^3mn^4a^2 & \\
+\ 48l^2m^4na^4 & -\ 64l^2m^3n^3a^2 & -\ 128l^4mn^4 + E^6. \\
-\ 1664l^2mn^4a^4 & & \\
+\ 192lm^3n^3a^4 & & \\
-\ 8m^5n^2a^4 & &
\end{array}
\right\}
$$

or to the compound quantity

$$
\left\{
\begin{array}{lll}
384m^2n^2a^{10} & +\ 168m^4na^8 & -\ 252lm^4na^6 \\
 & -\ 1152lm^2n^2a^8 & +\ 15m^6a^6 \\
 & -\ 512mn^4a^8 & -\ 320m^3n^3a^6 \\
 & & +\ 1248l^2m^2n^2a^6 \\
 & & +\ 1536lmn^4a^6 \\
+\ 84l^2m^4na^4 & -\ 160l^2m^3n^3a^2 & \\
+\ 480lm^3n^3a^4 & -\ 192lm^2n^5a^2 & \\
-\ 32m^5n^2a^4 & +\ 16m^4n^4a^2 & \\
+\ 128m^2n^5a^4 & +\ 96l^4m^2n^2a^2 & \\
-\ 576l^3m^2n^2a^4 & +\ 768l^3mn^4a^2 & \\
-\ 1664l^2mn^4a^4 & & +\ 64l^2m^2n^5 \\
 & & -\ 128l^4mn^4 + E^6.
\end{array}
\right\}
$$

Lastly,

Laſtly, ſince E² is $= 8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, and E⁴ is = the compound quantity

$$\left\{\begin{array}{llll} 64n^2a^8 & - 192ln^2a^6 & + 208l^2n^2a^4 & - 96l^3n^2a^2 \\ & + 16m^2na^6 & - 24lm^2na^4 & + 8l^2m^2na^2 \\ & & + m^4a^4 & + 16l^4n^2, \end{array}\right\}$$

we ſhall have E⁶ = to the product of the multiplication of this compound quantity into the quadrinomial quantity $8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, that is, to the compound quantity

$$\left\{\begin{array}{lll} 512n^3a^{12} & - 2304ln^3a^{10} & + 4224l^2n^3a^8 \\ & + 192m^2n^2a^{10} & - 576lm^2n^2a^8 \\ & & + 24m^4na^8 \\ - 4032l^3n^3a^6 & + 2112l^4n^3a^4 & - 576l^5n^3a^2 \\ + 624l^2m^2n^2a^6 & - 288l^3m^2n^2a^4 & + 48l^4m^2n^2a^2 \\ - 36lm^4na^6 & + 12l^2m^4na^4 & \\ + m^6a^6 & & + 64l^6n^3. \end{array}\right\}$$

Therefore the compound quantity

$$\left\{\begin{array}{lll} 9m^4a^4E^2 & + 6m^2a^2E^4 & \\ - 24m^3n^2a^2E^2 & - 8mn^2E^4 & + E^6 \\ + 16m^2n^4E^2 & & \end{array}\right\}$$

will be equal to the following very numerous compound quantity, to wit,

$$\begin{array}{lll}
512n^3a^{12} & -2304ln^3a^{10} & +4224l^2n^3a^8 \\
 & +192m^2n^2a^{10} & -576lm^2n^2a^8 \\
 & & +24m^4na^8 \\
 & +384m^2n^2a^{10} & +168m^4na^8 \\
 & & -1152lm^2n^2a^8 \\
 & & -512mn^4a^8
\end{array}$$

$$\begin{array}{lll}
-4032l^3n^3a^6 & +2112l^4n^3a^4 & -576l^5n^3a^2 \\
+624l^2m^2n^2a^6 & -288l^3m^2n^2a^4 & +48l^4m^2n^2a^2 \\
-36lm^4na^6 & +12l^2m^4na^4 & \\
+m^6a^6 & & \\
-252lm^4na^6 & +84l^2m^4na^4 & -160l^2m^3n^3a^2 \\
+15m^6a^6 & +480lm^3n^3a^4 & -192lm^2n^5a^2 \\
-320m^3n^3a^6 & -32m^5n^2a^4 & +16m^4n^4a^2 \\
 & +128m^2n^5a^4 & \\
+1248l^2m^2n^2a^6 & -576l^3m^2n^2a^4 & +96l^4m^2n^2a^2 \\
+1536lmn^4a^6 & -1664l^2mn^4a^4 & +768l^3mn^4a^2 \\
 & & +64l^6n^3 \\
 & & +64l^2m^2n^5 \\
 & & -128l^4mn^4,
\end{array}$$

or to the compound quantity

$$\begin{array}{llll}
512n^3a^{12} & -2304ln^3a^{10} & +4224l^2n^3a^8 & -4032l^3n^3a^6 \\
 & +576m^2n^2a^{10} & -1728lm^2n^2a^8 & +1872l^2m^2n^2a^6 \\
 & & +192m^4na^8 & -288lm^4na^6 \\
 & & -512mn^4a^8 & +16m^6a^6 \\
 & & & -320m^3n^3a^6 \\
 & & & +1536lmn^4a^6
\end{array}$$

$$\begin{array}{lll}
+2112l^4n^3a^4 & -576l^5n^3a^2 & +64l^6n^3 \\
-864l^3m^2n^2a^4 & +144l^4m^2n^2a^2 & +64l^2m^2n^5 \\
+96l^2m^4na^4 & -160l^2m^3n^3a^2 & -128l^4mn^4. \\
+480lm^3n^3a^4 & -192lm^2n^5a^2 & \\
-32m^5n^2a^4 & +16m^4n^4a^2 & \\
+128m^2n^5a^4 & +768l^3mn^4a^2. & \\
-1664l^2mn^4a^4 & &
\end{array}$$

Therefore

Therefore the compound quantity which forms the first, or left-hand, side of the equation set down at the end of art. 9, to wit, the compound quantity

$$\left\{\begin{array}{llll} 576m^2n^2a^{10} & -\ 1728lm^2n^2a^8 & +\ 1872l^2m^2n^2a^6 & \\ & +\ 192m^4na^8 & +\ 960lmn^4a^6 & \\ & -\ 384mn^4a^8 & -\ 256m^3n^3a^6 & \\ & & -\ 288lm^4na^6 & \\ & & +\ 16m^6a^6 & \\ & & +\ 64n^6a^6 & \\ -\ 864l^3m^2n^2a^4 & +\ 144l^4m^2n^2a^2 & & \\ -\ 768l^2mn^4a^4 & +\ 192l^3mn^4a^2 & & \\ +\ 352lm^3n^3a^4 & -\ 96l^2m^3n^3a^2 & & \\ +\ 96l^2m^4na^4 & +\ 64l^2n^6a^2 & & \\ -\ 32m^5n^2a^4 & -\ 64lm^2n^5a^2 & & \\ -\ 128ln^6a^4 & +\ 16m^3n^4a^2 & & \\ +\ 64m^2n^5a^4 & & & \end{array}\right\}$$

(which is equal to the compound quantity

$$\left\{\begin{array}{lll} 9m^4a^4E^2 & +\ 6m^2a^2E^4 & \\ -\ 24m^3n^2a^2E^2 & -\ 8mn^2E^4 & +\ E^6 \\ +\ 16m^2n^4E^2 & & \end{array}\right.$$

) will be equal to the compound quantity

$512n^3a^{12}$

$$\left.\begin{array}{llll}
512n^3a^{12} & -\ 2304ln^3a^{10} & +\ 4224l^2n^3a^8 & \\
 & +\ 576m^2n^2a^{10} & -\ 1728lm^2n^2a^8 & \\
 & & +\ 192m^4na^8 & \\
 & & -\ 512mn^4a^8 & \\
-\ 4032l^3n^3a^6 & +\ 2112l^4n^3a^4 & & \\
+\ 1872l^2m^2n^2a^6 & -\ 864l^3m^2n^2a^4 & & \\
-\ 288lm^4n\,a^6 & +\ 96l^2m^4n\,a^4 & & \\
+\ 16m^6a^6 & +\ 480lm^3n^3a^4 & & \\
-\ 320m^3n^3a^6 & -\ 32m^5n^2a^4 & & \\
+\ 1536lmn^4a^6 & +\ 128m^2n^5a^4 & & \\
 & -\ 1664l^2mn^4a^4 & & \\
 & -\ 576l^5n^3a^2 & & \\
 & +\ 144l^4m^2n^2a^2 & & \\
 & -\ 160l^2m^3n^3a^2 & & \\
 & -\ 192lm^2n^5a^2 & +\ 64l^6n^3 & \\
 & +\ 16m^4n^4a^2 & +\ 64l^2m^2n^5 & \\
 & +\ 768l^3mn^4a^2 & -\ 128l^4mn^4. &
\end{array}\right\}$$

Art. 11. Now the following eleven terms occur in both these compound quantities, and with the same signs + and — prefixed to them, to wit, the terms

$$\begin{array}{lll}
+\ 576m^2n^2a^{10}, & -\ 1728lm^2n^2a^8, & +\ 1872l^2m^2n^2a^6, \\
 & +\ 192m^4n\,a^8, & -\ 288lm^4n\,a^6, \\
 & & +\ 16m^6a^6, \\
-\ 864l^3m^2n^2a^4, & +\ 144l^4m^2n^2a^2, & \\
+\ 96l^2m^4n\,a^4, & +\ 16m^4n^4a^2. & \\
-\ 32m^5n^2a^4, & &
\end{array}$$

Therefore, if these terms are omitted, the remaining terms of these two compound quantities will still be equal to each other; that is, the compound quantity

—

$$\left\{\begin{array}{llll}
-\ 384mn^4a^8 & +\ 960lmn^4a^6 & -\ 768l^2mn^4a^4 \\
 & -\ 256m^3n^3a^6 & +\ 352lm^3n^3a^4 \\
 & +\ 64n^6a^6 & -\ 128ln^6a^4 \\
 & & +\ 64m^2n^5a^4 \\
 & \multicolumn{2}{l}{+\ 192l^3mn^4a^2} \\
 & \multicolumn{2}{l}{-\ 96l^2m^3n^3a^2} \\
 & \multicolumn{2}{l}{+\ 64l^2n^6a^2} \\
 & \multicolumn{2}{l}{-\ 64lm^2n^5a^2}
\end{array}\right\}$$

will be equal to the compound quantity

$$\left\{\begin{array}{lll}
512n^3a^{12} & -\ 2304ln^3a^{10} & +\ 4224l^2n^3a^8 \\
 & & -\ 512mn^4a^8 \\
-\ 4032l^3n^3a^6 & +\ 2112l^4n^3a^4 & -\ 576l^5n^3a^2 \\
-\ 320m^3n^3a^6 & +\ 480lm^3n^3a^4 & -\ 160l^2m^3n^3a^2 \\
+\ 1536lmn^4a^6 & +\ 128m^2n^5a^4 & -\ 192lm^2n^5a^2 \\
 & -\ 1664l^2mn^4a^4 & +\ 768l^3mn^4a^2 \\
 & \multicolumn{2}{l}{+\ 64l^6n^3} \\
 & \multicolumn{2}{l}{+\ 64l^2m^2n^5} \\
 & \multicolumn{2}{l}{-\ 128l^4mn^4.}
\end{array}\right\}$$

Add $512mn^4a^9 + 320m^3n^3a^6 + 1664l^2mn^4a^4 + 160l^2m^3n^3a^2 + 192lm^2n^5a^2$ to both ſides. And we ſhall then have the compound quantity

$$\left\{\begin{array}{llll} 128mn^4a^8 & +\ 960lmn^4a^6 & +\ 896l^2mn^4a^4 & \\ & +\ 64m^3n^3a^6 & +\ 352lm^3n^3a^4 & \\ & +\ 64n^6a^6 & -\ 128ln^6a^4 & \\ & & +\ 64m^2n^5a^4 & \\ +\ 192l^3mn^4a^2 & & & \\ +\ 64l^2m^3n^3a^2 & & & \\ +\ 64l^2n^6a^2 & & & \\ +\ 128lm^2n^5a^2 & & & \end{array}\right\}$$

= the compound quantity

$$\left\{\begin{array}{llll} 512n^3a^{12} & -\ 2304ln^3a^{10} & +\ 4224l^2n^3a^8 & -\ 4032l^3n^3a^6 \\ & & & +\ 1536lmn^4a^6 \\ +\ 2112l^4n^3a^4 & -\ 576l^5n^3a^2 & & \\ +\ 480lm^3n^3a^4 & +\ 768l^3mn^4a^2 & +\ 64l^6n^3 & \\ +\ 128m^2n^5a^4 & & +\ 64l^2m^2n^5 & \\ & & -\ 128l^4mn^4. & \end{array}\right\}$$

Now let $960lmn^4a^6 + 352lm^3n^3a^4 + 64m^2n^5a^4 + 192l^3mn^4a^2$ be fubtracted from both fides. And we fhall then have the compound quantity

$$\left\{\begin{array}{llll} 128mn^4a^8 & +\ 64m^3n^3a^6 & +\ 896l^2mn^4a^4 & +\ 64l^2m^3n^3a^2 \\ & +\ 64n^6a^6 & -\ 128ln^6a^4 & +\ 64l^2n^6a^2 \\ & & & +\ 128lm^2n^5a^2 \end{array}\right\}$$

= the compound quantity

$$\left\{\begin{array}{llll}
512n^3a^{12} & -\ 2304ln^3a^{10} & +\ 4224l^2n^3a^8 & \\
 & -\ 4032l^3n^3a^6 & +\ 2112l^4n^3a^4 & \\
 & +\ 576lmn^4a^6 & +\ 128lm^3n^3a^4 & \\
 & & +\ 64m^2n^5a^4 & \\
 & -\ 576l^5n^3a^2 & & \\
 & +\ 576l^3mn^4a^2 & +\ 64l^6n^3 & \\
 & & +\ 64l^2m^2n^5 & \\
 & & -\ 128l^4mn^4; &
\end{array}\right\}$$

and (dividing all the terms by n^3,) the compound quantity

$$\left\{\begin{array}{lll}
128mna^8 & +\ 64m^3a^6 & +\ 896l^2mna^4 \\
 & +\ 64n^3a^6 & -\ 128ln^3a^4 \\
 & +\ 64l^2m^3a^2 & \\
 & +\ 64l^2n^3a^2 & \\
 & +\ 128lm^2n^2a^2 &
\end{array}\right\}$$

will be equal to the compound quantity

$$\left\{\begin{array}{llll}
512a^{12} & -\ 2304la^{10} & +\ 4224l^2a^8 & -\ 4032l^3a^6 \\
 & & & +\ 576lmna^6 \\
+\ 2112l^4a^4 & -\ 576l^5a^2 & & \\
+\ 128lm^3a^4 & +\ 576l^3mna^2 & +\ 64l^6 & \\
+\ 64m^2n^2a^4 & & +\ 64l^2m^2n^2 & \\
 & & -\ 128l^4mn; &
\end{array}\right\}$$

and (dividing all the terms by 32,) the compound quantity

$$\left\{\begin{array}{llll}
4mna^8 & +\,2m^3a^6 & +\,28l^2mna^4 & \\
 & +\,2n^3a^6 & -\,4ln^3a^4 & \\
 & & +\,2l^2m^3a^2 & \\
 & & +\,2l^2n^3a^2 & \\
 & & +\,4l^4mna^2 &
\end{array}\right\}$$

will be equal to the compound quantity

$$\left\{\begin{array}{llll}
16a^{12} & -\,72la^{10} & +\,132l^2a^8 & -\,126l^3a^6 \\
 & & & +\,18lmna^6 \\
 & +\,66l^4a^4 & -\,18l^5a^2 & +\,2l^6 \\
 & +\,4lm^3a^4 & +\,18l^3mna^2 & +\,2l^2m^2n^2 \\
 & +\,2m^2n^2a^4 & & -\,4l^4mn
\end{array}\right\};$$

and (ſubtracting the former of theſe compound quantities from the latter,) we ſhall have the compound quantity

$$\left\{\begin{array}{llll}
16a^{12} & -\,72la^{10} & +\,132l^2a^8 & -\,126l^3a^6 \\
 & & & +\,18lmna^6 \\
 & & -\,4mna^8 & -\,2m^3a^6 \\
 & & & -\,2n^3a^6 \\
+\,66l^4a^4 & & & \\
+\,4lm^3a^4 & -\,18l^5a^2 & & \\
+\,2m^2n^2a^4 & +\,18l^3mna^2 & +\,2l^6 & \\
-\,28l^2mna^4 & -\,2l^2m^3a^2 & +\,2l^2m^2n^2 & \\
+\,4ln^3a^4 & -\,2l^2n^3a^2 & -\,4l^4mn & \\
 & -\,4lm^2n^2a^2 & &
\end{array}\right\}$$

$= 0.$

Art. 12.

Art. 12. Now let all the terms of this equation be divided by 16, in order to free the higheſt power of the unknown quantity a from it's co-efficient. And we ſhall then have

$$\left.\begin{array}{l}
a^{12} - \frac{9}{2}la^{10} + \frac{33}{4}l^2a^8 - \frac{63}{8}l^3a^6 + \frac{33}{8}l^4a^4 \\
\qquad - \frac{1}{4}mna^8 + \frac{9}{8}lmna^6 + \frac{1}{4}lm^3a^4 \\
\qquad\qquad - \frac{1}{8}m^3a^6 + \frac{1}{8}m^2n^2a^4 \\
\qquad\qquad - \frac{1}{8}n^3a^6 - \frac{7}{4}l^2mna^4 \\
\qquad\qquad\qquad + \frac{1}{4}ln^3a^4 \\
\quad - \frac{9}{8}l^5a^2 \qquad + \frac{1}{8}l^6 \\
\quad + \frac{9}{8}l^3mna^2 \qquad + \frac{1}{8}l^2m^2n^2 \\
\quad - \frac{1}{8}l^2m^3a^2 \qquad - \frac{1}{4}l^4mn \\
\quad - \frac{1}{8}l^2n^3a^2 \\
\quad - \frac{1}{4}lm^2n^2a^2
\end{array}\right\}$$

$$= 0.$$

This laſt equation may be free'd from fractions by taking $ee = 2aa$, or $\frac{ee}{2} = aa$, and ſubſtituting $\frac{ee}{2}$

inſtead

instead of aa in it's terms. For, if $\frac{ee}{2}$ is $= aa$, we shall have $\frac{e^4}{4} = a^4$, and $\frac{e^6}{8} = a^6$, and $\frac{e^8}{16} = a^8$, and $\frac{e^{10}}{32} = a^{10}$, and $\frac{e^{12}}{64} = a^{12}$. And consequently, by this substitution of $\frac{ee}{2}$ instead of aa, the last equation will be converted into the following equation, to wit,

$$
\left.
\begin{array}{l}
\frac{e^{12}}{64} - \frac{9}{2} \times l \times \frac{e^{10}}{32} + \frac{33}{4} \times l^2 \times \frac{e^8}{16} - \frac{63}{8} \times l^3 \times \frac{e^6}{8} \\
\qquad\qquad - \frac{1}{4} \times mn \times \frac{e^8}{16} + \frac{9}{8} \times lmn \times \frac{e^6}{8} \\
\qquad\qquad\qquad\qquad - \frac{1}{8} \times m^3 \times \frac{e^6}{8} \\
\qquad\qquad\qquad\qquad - \frac{1}{8}\, n^3 \times \frac{e^6}{8} \\
+ \frac{33}{8} \times l^4 \times \frac{e^4}{4} - \frac{9}{8} \times l^5 \times \frac{e^2}{2} \\
+ \frac{1}{4} \times lm^3 \times \frac{e^4}{4} + \frac{9}{8} \times l^3mn \times \frac{e^2}{2} + \frac{1}{8}\, l^6 \\
+ \frac{1}{8} \times m^2n^2 \times \frac{e^4}{4} - \frac{1}{8} \times l^2m^3 \times \frac{e^2}{2} + \frac{1}{8}\, l^2m^2n^2 \\
- \frac{7}{4} \times l^2mn \times \frac{e^4}{4} - \frac{1}{8} \times l^2n^3 \times \frac{e^2}{2} - \frac{1}{4}\, l^4mn \\
+ \frac{1}{4} \times ln^3 \times \frac{e^4}{4} - \frac{1}{4} \times lm^2n^2 \times \frac{e^2}{2}
\end{array}
\right\}
$$

$= 0$, or

$\frac{e^{12}}{64}$

$$\left\{\begin{array}{l}
\dfrac{e^{12}}{64} - \dfrac{9le^{10}}{64} + \dfrac{33l^2e^8}{64} - \dfrac{63l^3e^6}{64} + \dfrac{33l^4e^4}{32} \\
\qquad\qquad - \dfrac{mne^8}{64} + \dfrac{9lmne^6}{64} + \dfrac{lm^3e^4}{16} \\
\qquad\qquad\qquad\qquad - \dfrac{m^3e^6}{64} + \dfrac{m^2n^2e^4}{32} \\
\qquad\qquad\qquad\qquad - \dfrac{n^3e^6}{64} - \dfrac{7l^2mne^4}{16} \\
\qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{ln^3e^4}{16} \\
\quad - \dfrac{9l^5e^2}{16} \\
\quad + \dfrac{9l^3mne^2}{16} \qquad + \dfrac{1}{8}l^6 \\
\quad - \dfrac{l^2m^3e^2}{16} \qquad + \dfrac{1}{8}l^2m^2n^2 \\
\quad - \dfrac{l^2n^3e^2}{16} \qquad - \dfrac{1}{4}l^4mn \\
\quad + \dfrac{lm^2n^2e^2}{8}
\end{array}\right\}$$

$$= 0, \text{ or}$$

$$\left\{\begin{array}{llllllllll}
\frac{e^{12}}{64} & -\frac{9le^{10}}{64} & +\frac{33l^2e^8}{64} & -\frac{63l^3e^6}{64} & +\frac{66l^4e^4}{64} \\
 & & -\frac{mne^8}{64} & +\frac{9lmne^6}{64} & +\frac{4lm^3e^4}{64} \\
 & & & -\frac{m^3e^6}{64} & +\frac{2m^2n^2e^4}{64} \\
 & & & -\frac{n^3e^6}{64} & -\frac{28l^2mne^4}{64} \\
 & & & & +\frac{4ln^3e^4}{64} \\
 & -\frac{36l^5e^2}{64} & +\frac{8l^6}{64} & & \\
 & +\frac{36l^3mne^2}{64} & +\frac{8l^2m^2n^2}{64} & & \\
 & -\frac{4l^2m^3e^2}{64} & -\frac{16l^4mn}{64} & & \\
 & -\frac{4l^2n^3e^2}{64} & & & \\
 & -\frac{8lm^2n^2e^2}{64} & & &
\end{array}\right\}$$

$= 0$; and confequently (multiplying all the terms by 64) we fhall have

e^{12}

$$\left\{\begin{array}{lllll}
e^{12} & -\ 9le^{10} & +\ 33l^2e^8 & -\ 63l^3e^6 & +\ 66l^4e^4 \\
 & & -\ mne^8 & +\ 9lmne^6 & +\ 4lm^3e^4 \\
 & & & -\ m^3e^6 & +\ 2m^2n^2e^4 \\
 & & & -\ n^3e^6 & -\ 28l^2mne^4 \\
 & & & & +\ 4ln^3e^4 \\
 & -\ 36l^5e^2 & +\ 8l^6 & & \\
 & +\ 36l^3mne^2 & +\ 8l^2m^2n^2 & & \\
 & -\ 4l^2m^3e^2 & -\ 16l^4mn & & \\
 & -\ 4l^2n^3e^2 & & & \\
 & -\ 8lm^2n^2e^2 & & &
\end{array}\right\}$$

$$= 0.$$

A Reduction of the foregoing Equation of the Twelfth Order to an Equation of the Eighth Order by dividing it by the Trinomial Quantity $e^4 - 4le^2 + 4l^2$.

Art. 13. This equation may be reduced to an equation of the eighth power by dividing it by the trinomial quantity $e^4 - 4le^2 + 4l^2$, or the ſquare of $2l - e^2$; which Diviſion is ſet down by Dr. Wallis in his Algebra, chapter 61, and is as follows:

The

Therefore the quotient arifing from the divifion of the foregoing dividend, (which was the firft, or left-hand, fide of the laft equation, and was equal to o,) is the compound quantity

$$\left\{ \begin{array}{llllll} e^8 - 5le^6 & + 9l^2e^4 & - 7l^3e^2 & + 2l^4 \\ & - mne^4 & + 5lmne^2 & - 4l^2mn \\ & & - m^3e^2 & + 2m^2n^2. \\ & & - n^3e^2 & \end{array} \right\}$$

And confequently, as the faid dividend was equal to o, this laft compound quantity, which arifes from the divifion of it by the trinomial quantity $e^4 - 4le^2 + 4l^2$, muft be equal to o likewife; and therefore the final equation, by the refolution of which the value of *ee*, and confequently thofe of *aa* and *a* in Colonel Titus's Problem, is to be obtained, will be the equation

$$\left. \begin{array}{llllll} e^8 - 5le^6 & + 9l^2e^4 & - 7l^3e^2 & + 2l^4 \\ & - mne^4 & + 5lmne^2 & - 4l^2mn \\ & & - m^3e^2 & + 2m^2n^2 \\ & & - n^3e^2 & \end{array} \right\} = 0.$$

Art. 14. Now let the values of the three letters *l*, *m*, and *n*, be fubftituted, inftead of thofe letters themfelves, in the terms of this equation, to wit, 16 inftead of *l*, 17 inftead of *m*, and 18 inftead of *n*. And then we fhall have $5le^6$ ($= 5 \times 16 \times e^6$) $= 80e^6$; and $9l^2e^4$ ($= 9 \times 16 \times 16 \times e^4 = 9 \times 256 \times e^4$) $= 2304e^4$, and mne^4 ($= 17 \times 18 \times e^4$) $= 306e^4$, and confequently $9l^2e^4 - mne^4$ ($= 2304e^4 - 306e^4$) $= 1998e^4$; and

$7l^3e^2$

$7l^3e^2$ ($= 7 \times 16 \times 16 \times 16 \times e^2 = 7 \times 256 \times 16 \times e^2 = 7 \times 4096 \times e^2$) $= 28{,}672e^2$, and $5lmne^2$ ($= 5 \times 16 \times 17 \times 18 \times e^2 = 80 \times 17 \times 18 \times e^2 = 80 \times 306 \times e^2$) $= 24{,}480e^2$, and m^3e^2 ($= 17 \times 17 \times 17 \times e^2 = 289 \times 17 \times e^2$) $= 4913e^2$, and n^3e^2 ($= 18 \times 18 \times 18 \times e^2 = 324 \times 18 \times e^2$) $= 5832e^2$, and $7l^3e^2 + m^3e^2 + n^3e^2$ ($= 28{,}672e^2 + 4913e^2 + 5832e^2$) $= 39{,}417e^2$, and confequently $7l^3e^2 + m^3e^2 + n^3e^2 - 5lmne^2$ ($= 39{,}417e^2 - 24{,}480e^2 = 14{,}937e^2$; and, laftly, $2l^4$ ($= 2 \times l^3 \times l = 2 \times 4096 \times 16 = 2 \times 65{,}536$) $= 131{,}072$, and $2m^2n^2$ ($= 2 \times 17 \times 17 \times 18 \times 18 = 2 \times 289 \times 324 = 2 \times 93{,}636$) $= 187{,}272$, and $2l^4 + 2m^2n^2$ ($= 131{,}072 + 187{,}272$) $= 318{,}344$, and $4l^2mn$ ($= 4 \times 256 \times 17 \times 18 = 4 \times 256 \times 306 = 4 \times 78{,}336$) $= 313{,}344$, and confequently $2l^4 + 2m^2n^2 - 4l^2mn$ ($= 318{,}344 - 313{,}344$) $= 5000$. Therefore the equation

$$\left\{\begin{array}{llllll} e^8 - 5le^6 & + 9l^2e^4 & - 7l^3e^2 & + 2l^4 \\ & - mne^4 & + 5lmne^2 & - 4l^2mn \\ & & - m^3e^2 & + 2m^2n^2 \\ & & - n^3e^2 & \end{array}\right\}$$

$= 0$, will, when the values of l, m, n are fubftituted inftead of thofe letters in the feveral terms of the equation, and the feveral neceffary additions and fubtractions of the terms involving the fame powers of ee have been duly made, become $e^8 - 80e^6 + 1998e^4 - 14{,}937e^2 + 5000 = 0$; and therefore (adding $80e^6 + 14{,}937e^2$ to both fides) we fhall have $e^8 + 1998e^4 + 5000 =$

$14{,}937e^2$

$14,937e^2 + 80e^6$; and (ſubtracting $e^8 + 1998e^4$ from both ſides,) we ſhall have $5000 = 14,937e^2 + 80e^6 - e^8 - 1998e^4$, or $14,937e^2 - 1998e^4 + 80e^6 - e^8 = 5000$. And laſtly, ſubſtituting x inſtead of ee, we ſhall have the equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$, which has been reſolved by Dr. Halley in the foregoing tract, and more fully in the Appendix to the ſaid tract, in which all it's four roots have been inveſtigated, and have been ſhewn to be 0.350,987,046, and 12.756,441,794,480,744,02, &c, and 32.060,290,8, and 34.832,280,2.

Another Method of reducing the foregoing Equation of the Twelfth Order to an Equation of the Eighth Order by means of two Diviſions by the ſame Diviſor as before.

Art. 15. The foregoing reduction of the equation of the twelfth power to an equation of the eighth power by means of the diviſion of it by the trinomial quantity $e^4 - 4le^2 + 4l^2$ that is ſet forth in art. 13, may, perhaps, be thought to be in ſome degree obſcure and unſatisfactory, becauſe the dividend is a ſet of quantities that, all taken together, are equal to nothing; and it may be ſaid, with ſome appearance of truth, " that operations performed upon nothing, or a non-entity, can lead to no rational concluſion." I will therefore now ſeparate the quantities that compoſe that dividend, and place ſome of them on the ſecond, or right-hand, ſide of the equation, and then divide

divide both ſides of the equation ſo prepared by the ſame trinomial quantity $e^4 - 4le^2 + 4l^2$ as before; by which means we ſhall obtain two quotients which will be,—not equal to nothing, as the former quotient was, but,—of finite magnitudes and equal to each other: and then, I apprehend, no doubt can be entertained of the juſtneſs of the reaſoning and the truth of the concluſion obtained by it. This may be done in the following manner:

The compound quantity

$$\left\{\begin{array}{llllll}
e^{12} - 9le^{10} & + 33l^2e^8 & - 63l^3e^6 & + 66l^4e^4 & - 36l^5e^2 \\
 & - mne^8 & + 9lmne^6 & + 4lm^3e^4 & + 36l^3mne^2 \\
 & & - m^3e^6 & + 2m^2n^2e^4 & - 4l^2m^3e^2 \\
 & & - n^3e^6 & - 28l^2mne^4 & - 4l^2n^3e^2 \\
 & & & + 4ln^3e^4 & - 8lm^2n^2e^2 \\
 & & & \quad + 8l^6 & \\
 & & & \quad + 8l^2m^2n^2 & \\
 & & & \quad - 16l^4mn &
\end{array}\right\}$$

is equal to the trinomial quantity $e^{12} - 4le^{10} + 4l^2e^8$ together with the multinomial quantity

$$\left\{\begin{array}{llllll}
- 5le^{10} & + 29l^2e^8 & - 63l^3e^6 & + 66l^4e^4 & - 36l^5e^2 \\
 & - mne^8 & + 9lmne^6 & + 4lm^3e^4 & + 36l^3mne^2 \\
 & & - m^3e^6 & + 2m^2n^2e^4 & - 4l^2m^3e^2 \\
 & & - n^3e^6 & - 28l^2mne^4 & - 4l^2n^3e^2 \\
 & & & + 4ln^3e^4 & - 8lm^2n^2e^2 \\
 & & & \quad + 8l^6 & \\
 & & & \quad + 8l^2m^2n^2 & \\
 & & & \quad - 16l^4mn. &
\end{array}\right.$$

3 Therefore

Therefore the trinomial quantity $e^{12} - 4le^{10} + 4l^2e^8$ will be equal to the ſaid multinomial quantity when the ſigns of all it's terms ſhall be changed, that is, to the following multinomial quantity, to wit,

$$\left\{\begin{array}{llllllllll}
+ 5le^{10} & - 29l^2e^8 & + 63l^3e^6 & & - 66l^4e^4 & & + 36l^5e^2 \\
& + mne^8 & - 9lmne^6 & - & 4lm^3e^4 & - & 36l^3mne^2 \\
& & + m^3e^6 & - & 2m^2n^2e^4 & + & 4l^2m^3e^2 \\
& & + n^3e^6 & + & 28l^2mne^4 & + & 4l^2n^3e^2 \\
& & & - & 4ln^3e^4 & + & 8lm^2n^2e^2 \\
& & & & & - & 8l^6 \\
& & & & & - & 8l^2m^2n^2 \\
& & & & & + & 16l^4mn.
\end{array}\right.$$

For otherwiſe the former multinomial quantity, which conſiſted of the trinomial quantity $e^{12} - 4le^{10} + 4l^2e^8$ with this latter multinomial quantity ſubtracted from it, could not have been equal to nothing.

Now let both ſides of this equation be divided by the trinomial quantity $e^4 - 4le^2 + 4l^2$: and it will follow that the quotients of theſe two diviſions muſt be equal to each other.

But the quotient of the diviſion of the trinomial quantity $e^{12} - 4le^{10} + 4l^2e^8$ by the trinomial quantity $e^4 - 4le^2 + 4l^2$ is e^8. And the quotient of the diviſion of the ſecond, or right-hand, ſide of this equation, or of the laſt-mentioned multinomial quantity, may be found in the following manner;

The

The Division of the Compound Quantity, which forms the second, or right-hand, Side of the last Equation, by the Trinomial Quantity $e^4 - 4le^2 + 4l^2$.

The Divisor. The Dividend. The Quotient.

$$
e^4 - 4le^2 + 4l^2)\;
\begin{array}{llllll}
5le^{10} & -29l^2e^8 & +63l^3e^6 & -66l^4e^4 & +36l^5e^2 & \\
 & +mne^8 & -9lmne^6 & -4lm^3e^4 & -36l^3mne^2 & -8l^6 \\
 & & +m^3e^6 & -2m^2n^2e^4 & +4l^2m^3e^2 & -8l^2m^2n^2 \\
 & & +n^3e^6 & +28l^2mne^4 & +4l^2n^3e^2 & +16l^4mn \\
 & & & -4ln^3e^4 & +8lm^2n^2e^2 &
\end{array}
\left(\begin{array}{ll}
+5le^6 & -9l^2e^4 \\
 & +mne^4 \\
+7l^3e^2 & -2l^4 \\
-5lmne^2 & +4l^2mn \\
+m^3e^2 & -2m^2n^2 \\
+n^3e^2 &
\end{array}\right.
$$

The First Subtrahend. $5le^{10} - 20l^2e^8 + 20l^3e^6$

$$
\begin{array}{lllll}
* & -9l^2e^8 & +43l^3e^6 & -66l^4e^4 \\
 & +mne^8 & -9lmne^6 & -4lm^3e^4 \\
 & & +m^3e^6 & -2m^2n^2e^4 \\
 & & +n^3e^6 & +28l^2mne^4 \\
 & & & -4ln^3e^4
\end{array}
$$

The Second Subtrahend.

$$
\begin{array}{lll}
-9l^2e^8 & +36l^3e^6 & -36l^4e^4 \\
+mne^8 & -4lmne^6 & +4l^2mne^4
\end{array}
$$

$$\begin{array}{lllll}
* & +7l^3e^6 & -30l^4e^4 & +36l^5e^2 & \\
 & -5lmne^6 & +24l^2mne^4 & -36l^3mne^2 & \\
 & +m^3e^6 & -4lm^3e^4 & +4l^2m^3e^2 & \\
 & +n^3e^6 & -4ln^3e^4 & +4l^2n^3e^2 & \\
 & & -2m^2n^2e^4 & +8lm^2n^2e^2 & \\
\hline
\text{The Third Subtrahend.} & +7l^3e^6 & -28l^4e^4 & +28l^5e^2 & \\
 & -5lmne^6 & +20l^2mne^4 & -20l^3mne^2 & \\
 & +m^3e^6 & -4lm^3e^4 & +4l^2m^3e^2 & \\
 & +n^3e^6 & -4ln^3e^4 & +4l^3n^3e^2 & \\
\hline
 & * & -2l^4e^4 & +8l^5e^2 & -8l^6 \\
 & & +4l^2mne^4 & -16l^3mne^2 & -8l^2m^2n^2 \\
 & & -2m^2n^2e^4 & +8lm^2n^2e^2 & +16l^4mn \\
\hline
\text{The Fourth Subtrahend.} & & -2l^4e^4 & +8l^5e^2 & -8l^6 \\
 & & +4l^2mne^4 & -16l^3mne^2 & +16l^4mn \\
 & & -2m^2n^2e^4 & +8lm^2n^2e^2 & -8l^2m^2n^2 \\
\hline
 & & * & * & *
\end{array}$$

It

It appears therefore that the quotient of the divifion of this laft multinomial quantity by the trinomial quantity $e^4 - 4le^2 + 4l^2$ is the compound quantity

$$\left\{\begin{array}{llll} 5le^6 - 9l^2e^4 & + 7l^3e^2 & - 2l^4 \\ \quad + mne^4 & - 5lmne^2 & + 4l^2mn \\ & + m^3e^2 & - 2m^2n^2. \\ & + n^3e^2 & \end{array}\right\}.$$

Therefore this compound quantity will be equal to e^8, which is the quotient of the divifion of the quantity $e^{12} - 4le^{10} + 4l^2e^8$ by the fame divifor $e^4 - 4le^2 + 4l^2$; and therefore we have now obtained by intelligible means an equation between two finite quantities, to wit, the equation $e^8 =$

$$\left\{\begin{array}{llll} 5le^6 & - 9l^2e^4 & + 7le^2 & - 2l^4 \\ & + mne^4 & - 5lmne^2 & + 4l^2mn \\ & & + m^3e^2 & - 2m^2n^2, \\ & & + n^3e^2 & \end{array}\right.$$

which, by fubtracting the fecond, or right-hand, fide of it from e^8, or the left-hand fide of it, will produce the former equation obtained by Dr. Wallis, to wit, the equation

$$\left.\begin{array}{lllll} e^8 - 5le^6 & + 9l^2e^4 & - 7le^2 & + 2l^4 \\ & - mne^4 & + 5lmne^2 & - 4l^2mn \\ & & - m^3e^2 & + 2m^2n^2 \\ & & - n^3e^2 & \end{array}\right\} = 0.$$

The

The Application of the Number 12.756,441,794,480,744, *or the second Value of* ee, *or the least Value of it but one, in the Equation* $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$, *to the Solution of Colonel Titus's Problem.*

Art. 16. Having now at last, after many tedious and laborious Algebräick operations, obtained the equation

$$\left\{\begin{array}{lllll} e^8 - 5le^6 & + 9l^2e^4 & - 7le^2 & + 2l^4 \\ & - mne^4 & + 5lmne^2 & - 4l^2mn \\ & & - m^3e^2 & + 2m^2n^2 \\ & & - n^3e^2 & \end{array}\right\} = 0,$$

or (by substituting the values of l, m, and n in the terms of the equation instead of those letters themselves, and making the proper additions and subtractions of the terms that involve the same powers of ee) the equation $e^8 - 80e^6 + 1998e^4 - 14,937e^2 + 5000 = 0$, or the equation $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$, we will now proceed to apply one of the values of ee to the discovery of the value of aa, or $\frac{ee}{2}$, and of a, the first of the three unknown quantities a, b, and c, which are required to be found in Colonel Titus's Problem.

Now, if we suppose ee to be = 12.756,441,794, 480,744, (which is the second, or least but one, of the four values of ee in this equation that have been found in the foregoing Appendix,) this value of ee will enable us to find the values of the said three numbers a, b, and c required by the Problem.

For, if ee is = 12.756,441,794,480,744, we shall have aa (= $\frac{ee}{2}$ = $\frac{12.756,441,794,480,744}{2}$ = 6.378,220, 897,

897,240,372, and confequently a ($= \sqrt{6.378,220,897,240,372}$) $\doteq$ 2.525,513,986. Therefore a, or the firft and leaft of the three unknown quantities a, b, and c required to be found in the faid Problem, will be = 2.525,513,986. Q. E. I.

Having thus found the value of a, or the firft of the three numbers a, b, and c, we may derive from it the value of b, or the fecond of thofe numbers, in the manner following:

It has been fhewn above in art. 8 that b is = $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$.

Now aa is = 6.378,220, &c, and confequently a^4 will be ($= \overline{6.378,220, \&c}^2$) = 40.681,690,368,400, and $8na^4$ will be ($= 8 \times 18 \times a^4 = 144 \times a^4 = 144 \times 40.681,690,368,400$) = 5858.163,413,049,600; and m^2a^2 will be ($= 17 \times 17 \times a^2 = 289 \times a^2 = 289 \times 6.378,220$, &c) = 1843.305,580, &c; and $12lna^2$ will be ($= 12 \times 16 \times 18 \times a^2 = 3456 \times a^2 = 3456 \times 6.378,220$, &c) = 22,043.128,320; and $4l^2n$ will be ($= 4 \times 16 \times 16 \times 18 = 4 \times 256 \times 18 = 1024 \times 18$) = 18,432. Therefore the quadrinomial quantity $8na^4 - 12lna^2 + m^2a^2 + 4l^2n$ will be (= 5858.163,413,049,600 — 22,043.128,320 + 1843.305,580, &c + 18,432 = 26,133.468,993,049,600 — 22,043.128,320) = 4090.340,673,049,600. Therefore $\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}$ will be ($= \sqrt{4090.340,673,049,600}$) = 63.955,771; and $\frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$ will be ($= \frac{63.955,771}{2n}$ =

$= \frac{63.955,771}{2 \times 18} = \frac{63.955,771}{36}$) 1.776,549. And $\frac{ma}{2n}$ will be ($= \frac{17a}{36} = \frac{17 \times 2.525,513,986}{36} = \frac{42.933,737,762}{36}$) $=$ 1.192,603,826. Therefore $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$ will be (= 1.192,603, 826 + 1.776,549) = 2.969,152,826, or, nearly, 2.969, 153; that is, b, or the second of the unknown quantities a, b, and c, required to be found, will be =, nearly, 2.969,153. Q. E. I.

Lastly, since $aa + bc$ is $= l$, we shall have $bc = l - aa$, and consequently $c = \frac{l - aa}{b}$ ($= \frac{16 - 6.378,220,\&c}{2.969,153} = \frac{9.621,780}{2.969,153}$) = 3.240,580; that is, the third unknown number c which was required to be found will be = 3.240,580. Therefore the three numbers sought are 2.525,513,986, &c, or, nearly, 2.525,514, and 2.969,153, and 3.240,580. Q. E. I.

Art. 17. These numbers will answer the conditions of the problem. For, if a is = 2.525,514, and b is = 2.969,153, and c is = 3.240,580, we shall have aa = 6.378,220,8, and bb = 8.815,869,5, and cc = 10.501, 358,7, and bc (= 2.969,153 × 3.240,580) = 9.621, 778,8, and ac (= 2.525,514 × 3.240,580,) = 8.184, 130,1, and ab (= 2.525,514 × 2.969,153) = 7.498, 637,4. Therefore $aa + bc$ will be (= 6.378,220,8 + 9.621,778,8) = 15.999,998,6; which is very nearly equal to 16, agreeably to the first condition of the Problem.

blem. And $bb + ac$ will be (= 8.815,869,5 + 8.184,130,1) = 16.999,999,6; which is very nearly equal to 17, agreeably to the second condition of the Problem. And $cc + ab$ will be (= 10.501,358,7 + 7.498,637,4) = 17.999,996,1; which is very nearly equal to 18, agreeably to the third condition of the Problem. Q. E. D.

Art. 18. Dr. Wallis has investigated the values of the numbers a, b, and c to sixteen places of figures, and found them and their squares and products to be as follows; to wit,

a = 2.525,513,986,744,158, and aa = 6.378,220,897,240,372,

b = 2.969,152,768,619,848, and bb = 8.815,868,163,402,909,

c = 3.240,580,681,617,174, and cc = 10.501,363,154,070,430,

and bc = 9.621,779,102,759,628, and ac = 8.184,131,836,597,093,

and ab = 7.498,636,845,929,567; and consequently

$aa + bc$ to be =

$$\left\{ \begin{array}{rl} & 6.378,220,897,240,372, \\ + & 9.621,779,102,759,628 \end{array} \right\}$$

= 16.000,000,000,000,000, agreeably to the first condition of the Problem;

and $bb + ac$ to be =

$$\left\{ \begin{array}{rl} & 8.815,868,163,402,909 \\ + & 8.184,131,836,597,093 \end{array} \right\}$$

= 17.000,000,000,000,002, which is very

nearly

nearly equal to 17, agreeably to the second condition of the Problem;

and $cc + ab$ to be =

$$\left\{ \begin{array}{rl} & 10.501,363,154,070,430 \\ + & 7.498,636,845,929,567 \end{array} \right\}$$

= 17.999,999,999,999,997, which is very nearly equal to 18, agreeably to the third condition of the Problem.

The Application of the Number 0.350,987,045,866,14, *or the least Value of* ee *in the Equation* $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$, *to the Solution of Colonel Titus's Problem.*

Art. 19. But there is another set of values of the unknown numbers a, b, and c that will answer the conditions of the Problem. And these values may be derived from the least value of ee in the final equation $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$ resulting from the foregoing solution of the Problem. For the least value of ee in that equation, or of x in the equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$, has been found in the foregoing Appendix to Dr. Halley's Tract, page 160, to be = 0.350,986,9, or nearly, 0.350,987; and, if it had been investigated to a greater degree of exactness, it would have been found to be = 0.350,987,045,866,14. Therefore, if we take ee = 0.350,987,045,866,14, we shall have aa $\left(= \frac{ee}{2} = \frac{0.350,987,045,866,14}{2}\right) =$

0.175,

0.175,493,522,933.07, and confequently a ($= \sqrt{0.175,493,522,933,07}$) = 0.418,919,470,701,7; that is, the firſt and leaſt of the three numbers a, b, and c which are required to be found, will be = 0.418,919,470,701,7, or nearly, 0.418,919,470. Q. E. I.

Further, it has been ſhewn above in art. 8 that the ſecond unknown number b is $= \frac{ma}{2n} +$

$$\frac{\sqrt{8na^4 - 12nna^2 + mma^2 + 4l^2n}}{2n} = \frac{17a}{2 \times 18}$$

$$+ \frac{\sqrt{8 \times 18a^4 - 12 \times 16 \times 18a^2 + 17 \times 17a^2 + 4 \times 16 \times 16 \times 18}}{2 \times 18}$$

$$= \frac{17a}{36} + \frac{\sqrt{144a^4 - 12 \times 288a^2 + 289a^2 + 1024 \times 18}}{36}$$

$$= \frac{17a}{36} + \frac{\sqrt{144a^4 - 3456a^2 + 289a^2 + 18,432}}{36}$$

$$= \frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}.$$

But a is = 0.418,919,470, and aa is = 0.175,493,523, and confequently a^4 is ($= \overline{0.175,493,523}|^2$) = 0.030,797,976,63.

Therefore $144a^4$ will be (= 144 × 0.030,797,976,63) = 4.434,908,634,72, and $3167a^2$ will be (= 3167 × 0.175,493,523) = 555.787,987,341, and $144a^4 - 3167a^2 + 18,432$ will be (= 4.434,908,634,72 − 555.787,987,341 + 18,432 = 18,436.434,908,634,7 − 555.787,

$555.787,987,341$) $= 17,880.646,921,293,72$, and the square-root of $144a^4 - 3167a^2 + 18,432$ will be $(= \sqrt{17,880.646,921,293,72}) = 133.718,536,191$;

and $\frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}$ will be

$(= \frac{133.718,536,191}{36}) = 3.714,403,783$.

And $\frac{17a}{36}$ will be $(= \frac{17 \times 0.418,919,470}{36} = \frac{7.121,630,990}{36}) = 0.197,823,083$.

Therefore $\frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18432}}{36}$ will be $(= 0.197,823,083 + 3.714,403,783) = 3.912,226,866$; that is, b, or the fecond unknown quantity required to be found, will be $= 3.912,226,866$. Q. E. I.

Laftly, the third number c will be $= \frac{l - aa}{b}$ $(= \frac{16 - 0.175,493,523}{3.912,226,866} = \frac{15.824,506,477}{3.912,226,866}) = 4.044,884,670$.

Q. E. I.

Art. 20. And thefe three numbers, 0.418,919,470, 3.912,226,866, and 4.044,884,670, will anfwer the conditions of the Problem. For, if a is $= 0.418,919,470$, and b is $= 3.912,226,866$, and c is $= 4.044,884,670$, we fhall have

aa (=

$aa\ (= \overline{0.418,919,470}|^2) = 0.175,493,523,$
and $bb\ (= \overline{3.912,226,866}|^2) = 15.305,519,051,$
and $cc\ (= \overline{4.044,884,670}|^2) = 16.361,091,993,$
and $bc\ (= 3.912,226,866 \times 4.044,884,670) = 15.824,506,475,$
and $ac\ (= 0.418,919,470 \times 4.044,884,670) = 1.694,480,942,$
and $ab\ (= 0.418,919,470 \times 3.912,226,866) = 1.638,908,005.$

Therefore $aa + bc$ will be $=$

$$\left\{ \begin{array}{r} 0.175,493,523 \\ +\ 15.824,506,475 \end{array} \right\}$$

$= 15.999,999,998$; which is very nearly $= 16$, agreeably to the first condition of the Problem:

And $bb + ac$ will be $=$

$$\left\{ \begin{array}{r} 15.305,519,051 \\ +\ 1.694,480,942 \end{array} \right\}$$

$= 16.999,999,993$; which is very nearly equal to 17, agreeably to the second condition of the Problem:

And $cc + ab$ will be $=$

16.361,

$$\left\{ \begin{array}{r} 16.361,091,993 \\ +\ 1.638,908,005 \end{array} \right\}$$

$= 17.999,999,998$; which is very nearly equal to 18, agreeably to the third condition of the Problem.

It appears therefore that theſe three numbers 0.418, 919,470, 3.912,226,866, and 4.044,884,670, which are derived from 0.350,987,046, or the firſt, or leaſt, value of ee in the equation $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$, anſwer the conditions of the Problem as well as the three former numbers 2.525,513,986, and 2.969,153, and 3.240,580, which are derived from 12.756,441,794, 480,744, or the ſecond value of ee, or the leaſt value of ee but one, in the ſame equation.

Of the Two Greateſt Values of ee *in the Equation* $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$.

Art. 21. But the other two values of ee in the equation $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$, to wit, the numbers 32.060,290,8 and 34.832,280,2, have no relation to the preſent Problem: for, if the values of a, b, and c are derived from them by taking $aa = \frac{ee}{2}$, upon a ſuppoſition that ee is equal to either of thoſe two numbers, and taking $b = \frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$, or $\frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}$, and $c = \frac{l - aa}{b}$, or $\frac{16 - aa}{b}$, the values of $aa + bc$, $bb + ac$, and

cc

$cc + ab$ will not be found to be refpectively equal to the three numbers 16, 17, and 18, agreeably to the conditions of the Problem. But thefe two greater values of ee in the faid equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$, or thefe two greateft roots of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, will be found to relate to two other Problems, fomewhat different from that of Colonel Titus. For, if ee is taken $=$ 32.060,290,8, and aa is taken $= \frac{ee}{2}$, or $\frac{32.060,290,8}{2}$, or 16.030,145,4, and $a = \sqrt{16.030,145,4}$, and b is taken $= \frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18{,}432}}{36}$, and $c = \frac{l - aa}{b}$, or $\frac{16 - aa}{b}$, the three values of a, b, and c fo obtained will be fitted to anfwer the conditions of a Problem in which it fhould be required to find the values of three unknown numbers a, b, and c, upon a fuppofition that $aa - bc$ (inftead of $aa + bc$) was equal to 16, and that $bb + ac$ was (as before) equal to 17, and that $cc - ab$ (inftead of $cc + ab$) was equal to 18. And, if ee is taken $=$ 34.832,280,2, and aa is taken $= \frac{ee}{2}$, or $\frac{34.832,280,2}{2}$, or 17.416,140,1, and b is taken $=$ $\frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18{,}432}}{36}$, and c is taken $= \frac{l - aa}{b}$, or $\frac{16 - aa}{b}$, the three values of a, b, and c fo obtained will be fitted to anfwer the conditions of a third

third Problem, in which it ſhould be required to find the values of three unknown numbers *a*, *b*, and *c*, upon a ſuppoſition that $aa - bc$ (inſtead of $aa + bc$) was equal to 16, and that $bb - ac$ (inſtead of $bb + ac$) was equal to 17, and that $cc + ab$ was (as before) equal to 18. See Dr. Wallis's Algebra, chapter LXII, articles 63, 64, 65, 66, &c - - 79, in which the ſubject is fully diſcuſſed, but not without ſome degree of obſcurity ariſing from the conſideration of negative quantities, and from the doctrine of the generation of equations one from another by multiplication, or by bringing all the terms of each equation to the firſt, or left-hand, ſide of the equation, ſo as to make them equal to 0, and then multiplying the equations (ſo prepared and made equal to nothing,) one into another, and from the eminently falſe poſition derived from that manner of generating equations, to wit, "that every Algebräick equation has as many roots as there are units in the index of the higheſt power of the unknown quantity contained in the equation." For this doctrine of the generation of equations one from another by multiplication, (which was invented by Harriot, and adopted by Des Cartes and Dr. Wallis and almoſt all the ſubſequent writers on Algebra,) inſtead of being an improvement in that ſcience, has, in my opinion, been of great detriment to it by deſtroying it's ſimplicity and perſpicuity, and therefore ought again to be diſcarded from it.

Art. 22. The foregoing Solution of Colonel Titus's Problem given us by Dr. Wallis is, as we have ſeen, exceedingly tedious and laborious; and great part of the

labour required in the ſolution of it ariſes from the neceſſity we are under of raiſing the equation involving the unknown quantity *a* to the 12th order, or to the 12th power of *a*, in order to free it's terms from radicality. But Mr. William Frend, the ingenious author of the late perſpicuous Treatiſe on Algebra in one volume octavo, intitled *Principles of Algebra*, (in which he totally rejects the abſurd and perplexing doctrine of *negative quantities*, or *quantities leſs than nothing*, or *quantities obtained by ſubtracting a greater quantity from a leſſer*,) has lately communicated to me another Solution of this Problem, which produces only a biquadratick equation. And this equation will be found to have three roots, or, in the language of modern Algebräiſts, three real and affirmative roots; of which the middle root will enable us to find the firſt ſet of values of the three unknown numbers *a*, *b*, and *c*, that will anſwer the conditions of the Problem, to wit, the three numbers 2.525,5 &c, 2.969,15 &c, and 3.240,5 &c; and the greateſt root will enable us to find the ſecond ſet of values of the ſaid three unknown numbers that will anſwer the ſame conditions, to wit, the three numbers 0.418,919,47, 3.912,226,8 &c, and 4.044,884,6 &c. This Solution I ſhall now proceed to lay before the reader, as a proper accompaniment to the foregoing Solution of this Problem given us by Dr. Wallis.

ANOTHER

ANOTHER SOLUTION

OF

COLONEL TITUS's PROBLEM,

BY MR. WILLIAM FREND, M. A.

FELLOW OF JESUS COLLEGE, CAMBRIDGE.

Art. 23. RETAINING the notation ufed in the foregoing Solution of this Problem by Dr. Wallis, and retaining likewife the preliminary obfervations made by Dr. Wallis, to wit, " that the three unknown numbers a, b, and c, that are fought in the Problem, are, all of them, of different magnitudes, and that a, or the number of which the fquare occurs in the firft equation $aa + bc = 16$, is the leaft of the three, and that c, or the number of which the fquare occurs in the third equation $cc + ab = 18$, is the greateft of the three," let it be fuppofed that the fecond unknown number b is greater than the firft unknown number a in the proportion of x to 1, and that the third unknown number c is greater than the firft unknown number in the proportion of y to 1.

Then will b be $= xa$, and c will be $= ya$, and confequently bb will be $= x^2a^2$, and cc will be $= y^2a^2$, and

bc

bc will be $(= xa \times ya) = xyaa$, and ac will be $(= a \times ya) = yaa$, and ab will be $(= a \times xa) = xaa$. Therefore the firft equation $aa + bc = 16$, or $aa + bc = l$, will now become $aa + xyaa = 16$, or $aa + xyaa = l$; and the fecond equation $bb + ac = 17$, or $bb + ac = m$, will now become $x^2a^2 + ya^2 = 17$, or $x^2a^2 + ya^2 = m$; and the third equation $cc + ab = 18$, or $cc + ab = n$, will now become $y^2a^2 + xa^2 = 18$, or $y^2a^2 + xa^2 = n$.

Art. 24. Now, fince $aa + xyaa$ is $= l$, we fhall have $aa = \frac{l}{1 + xy}$. And, fince $x^2a^2 + ya^2$ is $= m$, we fhall have $aa = \frac{m}{x^2 + y}$. And, fince $y^2a^2 + xa^2$ is $= n$, we fhall have $aa = \frac{n}{yy + x}$.

Further, fince aa is $= \frac{l}{1 + xy}$, and aa is alfo $= \frac{m}{x^2 + y}$, we fhall have $\frac{l}{1 + xy} = \frac{m}{x^2 + y}$; and confequently, (multiplying both fides by $\overline{1 + xy} \times \overline{x^2 + y}$) we fhall have $lx^2 + ly = m + mxy$; and (fubtracting m from both fides,) $lx^2 + ly - m = mxy$, and (fubtracting ly from both fides) $lx^2 - m = mxy - ly$, and (dividing both fides by $mx - l$) we fhall have $y = \frac{lx^2 - m}{mx - l}$. And thus we have obtained a value of y expreffed by a fraction which involves only the known

R quantities

quantities l and m and the unknown quantity x. We muſt now endeavour to find another value of y expreſſed, like the former, in terms that involve only known quantities and the ſame unknown quantity x. And this may be done as follows:

Art. 25. Since aa is $= \frac{m}{x^2 + y}$, and aa is likewiſe $= \frac{n}{yy + x}$, it follows that $\frac{m}{x^2 + y}$ will be $= \frac{n}{yy + x}$. Therefore (multiplying both ſides into $\overline{x^2 + y} \times \overline{y^2 + x}$) we ſhall have $myy + mx = nx^2 + ny$; and (ſubtracting mx from both ſides,) we ſhall have $myy = nxx - mx + ny$, and (ſubtracting ny from both ſides,) $myy - ny = nxx - mx$, and (dividing both ſides by m) $yy - \frac{ny}{m} = \frac{nxx - mx}{m}$. Therefore (adding $\frac{n^2}{4m^2}$ to both ſides,) we ſhall have $yy - \frac{ny}{m} + \frac{n^2}{4m^2} = \frac{nxx - mx}{m} + \frac{n^2}{4m^2}$; and (extracting the ſquare-roots of both ſides) we ſhall have $y - \frac{n}{2m} = \sqrt{\frac{nxx - mx}{m} + \frac{n^2}{4m^2}}$, and (adding $\frac{n}{2m}$ to both ſides,) $y =$

$$\sqrt{\frac{nxx - mx}{m} + \frac{n^2}{4m^2}} + \frac{n}{2m}.$$

Art. 26. But it was before ſhewn that y is $= \frac{lx^2 - m}{mx - l}$.

Therefore

Therefore $\frac{lx^2 - m}{mx - l}$ will be $= \sqrt{\frac{nxx - mx}{m} + \frac{n^2}{4m^2}}$ $+ \frac{n}{2m}$; and consequently $\frac{lx^2 - m}{mx - l} - \frac{n}{2m}$ will be $=$ $\sqrt{\frac{nxx - mx}{m} + \frac{n^2}{4m^2}}$, and therefore the square of $\frac{lx^2 - m}{mx - l} - \frac{n}{2m}$ will be equal to $\frac{nxx - mx}{m} + \frac{n^2}{4m^2}$.

But the square of $\frac{lx^2 - m}{mx - l} - \frac{n}{2m}$ is $= \left(\frac{lx^2 - m}{mx - l}\right)^2$ $+ \left(\frac{n}{2m}\right)^2 - 2 \times \frac{n}{2m} \times \frac{lx^2 - m}{mx - l} = \frac{(lx^2 - m)^2}{(mx - l)^2}$ $+ \frac{n^2}{4m^2} - \frac{n}{m} \times \frac{lx^2 - m}{mx - l} = \frac{l^2x^4 - 2lmx^2 + m^2}{(mx - l)^2}$ $+ \frac{n^2}{4m^2} - \frac{n}{m} \times \frac{lx^2 - m}{mx - l}$.

Therefore $\frac{l^2x^4 - 2lmx^2 + m^2}{(mx - l)^2} + \frac{n^2}{4m^2} - \frac{n}{m} \times$ $\frac{lx^2 - m}{mx - l}$ will be $= \frac{nxx - mx}{m} + \frac{n^2}{4m^2}$. And consequently (subtracting $\frac{n^2}{4m^2}$ from both sides,) we shall have $\frac{l^2x^4 - 2lmx^2 + m^2}{(mx - l)^2} - \frac{n}{m} \times \frac{lx^2 - m}{mx - l} =$ $\frac{nxx - mx}{m}$, and (multiplying both sides into m)

$$\frac{l^2mx^4 - 2lm^2x^2 + m^3}{\overline{mx - l}|^2} - n \times \frac{lx^2 - m}{mx - l} = nxx - mx;$$

and (multiplying both sides into $\overline{mx-l}|^2$, or $m^2x^2 - 2lmx + l^2$,) we shall have $l^2mx^4 - 2lm^2x^2 + m^3 - n \times \overline{lx^2 - m} \times \overline{mx - l} = \overline{nxx - mx} \times \overline{m^2x^2 - 2lmx + l^2}$, or $l^2mx^4 - 2lm^2x^2 + m^3 - n \times \overline{lmx^3 - l^2x^2 - m^2x + lm} = m^2nx^4 - 2lmnx^3 + l^2nx^2 - m^3x^3 + 2lm^2x^2 - l^2mx$, or $l^2mx^4 - 2lm^2x^2 + m^3 - lmnx^3 + l^2nx^2 + m^2nx - lmn = m^2nx^4 - 2lmnx^3 + l^2nx^2 - m^3x^3 + 2lm^2x^2 - l^2mx$, and (subtracting l^2nx^2 from both sides,) $l^2mx^4 - 2lm^2x^2 + m^3 - lmnx^3 + m^2nx - lmn = m^2nx^4 - 2lmnx^3 - m^3x^3 + 2lm^2x^2 - l^2mx$, and (adding $lmnx^3 + 2lm^2x^2$ to both sides) $l^2mx^4 + m^3 + m^2nx - lmn = m^2nx^4 - lmnx^3 - m^3x^3 + 4lm^2x^2 - l^2mx$, and (subtracting $l^2mx^4 + m^2nx$ from both sides,) $m^3 - lmn = m^2nx^4 - l^2mx^4 - lmnx^3 - m^3x^3 + 4lm^2x^2 - l^2mx - m^2nx$, or $m^2nx^4 - l^2mx^4 - lmnx^3 - m^3x^3 + 4lm^2x^2 - l^2mx - m^2nx = m^3 - lmn$, and (dividing all the terms by m) $mnx^4 - l^2x^4 - lnx^3 - m^2x^3 + 4lmx^2 - l^2x - mnx = m^2 - ln$, or $\overline{mn - l^2} \times x^4 - \overline{ln + m^2} \times x^3 + 4lmx^2 - \overline{l^2 + mn} \times x = m^2 - ln$; and, lastly, (dividing all the terms by $mn - l^2$) we shall have $x^4 - \frac{ln + m^2}{mn - l^2} \times x^3 + \frac{4lm}{mn - l^2} \times x^2 - \frac{l^2 + mn}{mn - l^2} \times x = \frac{m^2 - ln}{mn - l^2}$.

Art. 27. But, because l is $= 16$, and m is $= 17$, and n is $= 18$, we shall have ln $(= 16 \times 18) = 288$, and

mn

mn $(= 17 \times 18) = 306$, and l^2 $(= 16 \times 16) = 256$, and m^2 $(= 17 \times 17) = 289$. Therefore $\frac{ln + m^2}{mn - l^2}$ will be $\left(= \frac{288 + 289}{306 - 256} = \frac{577}{50} = \frac{1154}{100}\right) = 11.54$; and $\frac{4lm}{mn - l^2}$ will be $\left(= \frac{4 \times 16 \times 17}{306 - 256} = \frac{4 \times 272}{50} = \frac{1088}{50} = \frac{2176}{100}\right) = 21.76$; and $\frac{l^2 + mn}{mn - l^2}$ will be $\left(= \frac{256 + 306}{306 - 256} = \frac{562}{50} = \frac{1124}{100}\right) = 11.24$; and $\frac{m^2 - ln}{mn - l^2}$ will be $= \frac{289 - 288}{306 - 256} = \frac{1}{50} = \frac{2}{100}) = 0.02$.

Therefore the equation $x^4 - \overline{\frac{ln + m^2}{mn - l^2}} \times x^3 + \frac{4lm}{mn - l^2} \times x^2 - \overline{\frac{l^2 + mn}{mn - l^2}} \times x = \frac{m^2 - ln}{mn - l^2}$ will, when these substitutions of the original numbers 16, 17, and 18 for the letters l, m, and n have been made in it, become $x^4 - 11.54 \times x^3 + 21.76 \times x^2 - 11.24 \times x = 0.02$. This equation must therefore be resolved in order to obtain the value of x; and from the value of x so obtained, or, if the equation should have more than one root, (as will be found to be the case,) from that value of x which has a relation to the present Problem, we must then derive the value of y, expressed by it's relation to x, by computing the fraction $\frac{lx^2 - m}{mx - l}$, or $\frac{16x^2 - 17}{17x - 16}$, to which we have seen, in art. 24, that y is equal. And, when this is done, we must multiply the value of xa, or b, into the

value of ya, or c, which will give us the value of bc expressed by it's relation to aa; and then the equation $aa + bc = 16$ will be a quadratick equation involving only one unknown quantity, namely, a, and which may therefore be easily resolved; and the resolution of it will give us the value of a, or the first of the three unknown quantities a, b, and c, that the Problem requires us to find. And from the value of a so found we may derive the values of b and c by multiplying a into x to obtain the number b, and by multiplying a into y to obtain the number c. Our next business therefore must be to find the roots of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

The Resolution of the Biquadratick Equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Art. 28. Let us, first, suppose that x is $= 1$.

And we shall then have $x^2 = 1$, and $x^3 = 1$, and $x^4 = 1$; and consequently $x^4 + 21.76x^2$ will be $(= 1 + 21.76) = 22.76$, and $11.54x^3 + 11.24x$ will be $(= 11.54 + 11.24) = 22.78$; which is greater than 22.76, and therefore cannot be subtracted from it. Therefore in this case the binomial quantity $11.54x^3 + 11.24x$ is greater than the binomial quantity $x^4 + 21.76x^2$, and cannot be subtracted from it, and consequently the quadrinomial quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ (which

(which ſuppoſes the ſubtraction of $11.54x^3 + 11.24x$ from $x^4 + 21.76x^2$) cannot exiſt. And the ſame thing will be true, if x be of any magnitude leſs than 1. But, when x is a little greater than 1, the binomial quantity $x^4 + 21.76x^2$ (which before had been leſs than the binomial quantity $11.54x^3 + 11.24$) will become equal to the binomial quantity $11.54x^3 + 11.24x$, and the whole quadrinomial quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $= 0$; and, when x increaſes ſtill further, the ſaid quadrinomial quantity will increaſe from 0 to a certain quantity, after which it will decreaſe to a certain quantity, and then, finally, it will increaſe from that quantity *ad infinitum*, which makes it become at three different inſtants of time equal to the ſame quantity. Thus, for example, when x is become $= 1 + \frac{1}{10}$, or 1.1, we ſhall have $x^2 = 1.21$, and $x^3 = 1.331$, and $x^4 = 1.14641$, and $11.24x$ ($= 11.24 \times 1.1$) $= 12.364$, and $21.76x^2$ ($= 21.76 \times 1.21$) $= 26.3296$, and $11.54x^3$ ($= 11.54 \times 1.331$) $= 15.359,74$; and conſequently the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be ($= 1.4641 - 15.359,74 + 26.3296 - 12.364 = 27.793,70 - 27.723,74$) $= 0.069,96$. Therefore, while x increaſes from the magnitude which it has when $x^4 + 21.76x^2$ is $= 11.54x^3 + 11.24x$, or when the quadrinomial quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ is $= 0$, to 1.1, the ſaid quadrinomial quantity will increaſe from 0 to 0.069,96, and conſequently it will at ſome former inſtant of time during it's ſaid increaſe have been equal to 0.2, which is leſs than 0.069,96; or, in other words, the leaſt

root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ will be lefs than 1.1, but greater than 1.

Art. 29. Since we now know that the faid root is lefs than 1.1, but greater than 1, we will, in the next place, fuppofe it to be nearly = 1.03, and will try the effect of this fuppofition.

Now, if x is = 1.03, we fhall have xx ($= \overline{1.03}|^2$) = 1.0609, and x^3 ($= \overline{1.03}|^3$) = 1.092,727, and x^4 ($= \overline{1.03}|^4$) = 1.125,508,81. Therefore $11.24x$ will be ($= 11.24 \times 1.03$) = 11.5772; and $21.76x^2$ will be ($= 21.76 \times 1.0609$) = 23.085,184; and $11.54x^3$ will be ($= 11.54 \times 1.092,727$) = 12.610,069,58; and confequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be (= 1.125,508,81 − 12.610,069,58 + 23.085,184,00 − 11.577,200,00 = 24.210,692,81 − 24.187,269,58) = 0.023,423,23; which (though much lefs than 0.069,96, or the refult of the fubftitution of 1.1 inftead of x in the faid compound quantity) is ftill a little greater than 0.02, or the abfolute term of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. And confequently 1.03 will be fomewhat greater than the root of that equation.

Art. 30. We will therefore, in the third place, fuppofe x to be = 1.027 (inftead of 1.03, or 1.030) and try the effect of that conjecture.

Now,

Now, if x is $= 1.027$, we shall have x^2 $(= \overline{1.027}|^2) = 1.054,729$, and x^3 $(= \overline{1.027}|^3) = 1.083,206,683$, and x^4 $(= \overline{1.027}|^4) = 1.112,453,441$. Therefore $11.24x$ will be $(= 11.24 \times 1.027) = 11.543,48$, and $21.76x^2$ will be $(= 21.76 \times 1.054,729) = 22.950,903,04$, and $11.54x^3$ will be $(= 11.54 \times 1.083,206,683) = 12.500,205,121,82$; and consequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $(= 1.112,453,263,441 - 12.500,205,121,82 + 22.950,903,04 - 11.543,48 = 24.063,356,303,441 - 24.043,685,121,82) = 0.019,671,181,621$; which is a little less than 0.02, or the absolute term of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. Therefore 1.027 will be a little less than the true value of x in that equation. We may therefore now conclude that the said true value is less than 1.03, or 1.030, but greater than 1.027, and likewise that it is nearer to 1.027 than to 1.03, or 1.030.

Art. 31. We will now make use of this last near value of x, to wit, 1.027, as a basis for a further approach to it's true value by Mr. Raphson's method of approximation, and for that purpose we will suppose x to be equal to $1.027 + z$, and will substitute that binomial quantity instead of x in the terms of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. This may be done as follows:

If x is $= 1.027 + z$, we shall have x^2 $(= \overline{1.027 + z}|^2 = \overline{1.027}|^2 + 2 \times 1.027 \times z + \&c = \overline{1.027}|^2 + 2.054 \times z + \&c) = 1.054,729 + 2.054 \times z + \&c$,

and

and x^3 $(= \overline{1.027 + z}|^3 = \overline{1.027}|^3 + 3 \times \overline{1.027}|^2 \times z + \&c$
$= \overline{1.027}|^3 + 3 \times 1.054,729 \times z + \&c = \overline{1.027}|^3$
$+ 3.164,187 \times z + \&c) = 1.083,206,683 +$
$3.164,187 \times z + \&c,$

and x^4 $(= \overline{1.027 + z}|^4 = \overline{1.027}|^4 + 4 \times \overline{1.027}|^3 \times z + \&c$
$= \overline{1.027}|^4 + 4 \times 1.083,206,683 \times z + \&c =$
$\overline{1.027}|^4 + 4.332,826,732 \times z + \&c) = 1.112,453,$
$263,441 + 4.332,826,731 \times z + \&c.$

Therefore $11.24x$ will be $(= 11.24 \times \overline{1.027 + z} =$
$11.24 \times 1.027 + 11.24 \times z) = 11.543,48 + 11.24 \times z;$

And $21.76x^2$ will be $(= 21.76 \times$

$$\overline{1.054,729 + 2.054 \times z + \&c}$$

$$= 21.76 \times 1.054,729 + 21.76 \times 2.054 \times z + \&c$$
$$= 21.76 \times 1.054,729 + 44.695,04 \times z + \&c)$$
$$= 22.950,903,04 \quad + 44.695,04 \times z + \&c;$$

And $11.54x^3$ will be $(= 11.54 \times$

$$\overline{1.083,206,683 + 3.164,187 \times z + \&c}$$

$$= 11.54 \times 1.083,206,683 + 11.54 \times 3.164,187 \times z + \&c$$
$$= 11.54 \times 1.083,206,683 + 35.565,461,88 \times z + \&c)$$
$$= 12.500,205,121,82 \quad + 35.565,461,88 \times z + \&c;$$

and consequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $=$ to the compound quantity

$$\left\{\begin{array}{llll} & 1.112,453,263,441 & + \ 4.332,826,732 & \times z + \&c \\ - & 12.500,205,121,82 & - \ 35.565,461,88 & \times z - \&c \\ + & 22.950,903,04 & + \ 44.695,04 & \times z + \&c \\ - & 11.543,48 & - \ 11.24 & \times z \end{array}\right\}$$

$$= \left\{\begin{array}{llll} & 24.063,356,303,441 & + \ 49.027,866,732 & \times z + \&c \\ - & 24.043,685,121,82 & - \ 46.805,461,88 & \times z - \&c \end{array}\right\}$$

$$= 0.019,671,181,621 + 2.222,404,852 \times z + \&c.$$

But the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ is $= 0.02$.

Therefore the compound quantity $0.019,671,181,621 + 2.222,404,852 \times z$ will also be $= 0.02$. And consequently, (subtracting $0.019,671,181,621$ from both sides,) we shall have $2.222,404,852 \times z$ $(= 0.020,000,000 - 0.019,671,181) = 0.000,328,379$, and z $\left(= \frac{0.000,328,379}{2.222,404,852}\right) = 0.000,147,9$, or, nearly, $0.000,148$. Therefore x, or $1.027 + z$, will be $(= 1.027 + 0.000,148) = 1.027,148$; that is, the root of the equation $x^4 - 11.54x^3 + 21.76x^2 + 11.24x = 0.02$ will be, nearly, $= 1.027,148$. Q. E. I.

Art. 32. We will therefore, in the next place, suppose x to be $= 1.027,148$, and try the effect of this supposition.

Now,

Now, if x is $= 1.027,148$, we shall have x^2 ($= \overline{1.027,148})^2$) $= 1.055,033,013,904$, and x^3 ($= \overline{1.027,148}^3$ $= 1.055,033,0 \times 1.027,148$) $= 1.083,675,0$, and x^4 ($= 1.083,675,0 \times 1.027,148$) $= 1.113,194,6$. Therefore $11.24x$ will be ($= 11.24 \times 1.027,148$) $= 11.545,143,5$, and $21.76x^2$ will be ($= 21.76 \times 1.055,033,0$) $= 22.957,518,0$, and $11.54x^3$ will be ($= 11.54 \times 1.083,675,0$) $= 12.505,609,5$; and consequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be ($= 1.113,194,6 - 12.505,609,5 + 22.957,518,0 - 11.545,143,5 = 24.070,712,6 - 24.050,753,0$) $= 0.019,959,6$; which is a little less than 0.02, or the absolute term of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. Therefore 1.027,148 will be a little less than the true value of x in that equation.

Art. 33. We will now have recourse to a second process of Mr. Raphson's method of approximation, in order to obtain a more exact value of this root of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, and for that purpose will suppose x to be $= 1.027,148 + z$, and will substitute $1.027,148 + z$ instead of it in the terms of the said equation, and then resolve the transformed equation resulting from such substitution, as if it were a mere simple equation, by omitting all the terms of it that involve any higher power of the unknown quantity z than the simple power, or z itself. This may be done in the manner following:

Since

Since x is $= 1.027,148 + z$, we shall have x^2 ($= \overline{1.027,148 + z})^2 = \overline{1.027,148}|^2 + 2 \times 1.027,148 \times z + \&c = \overline{1.027,148})^2 + 2.054,296 \times z + \&c) = 1.055,033,013,904 + 2.054,296 \times z + \&c$,

and x^3 ($= \overline{1.027,148 + z}|^3 = \overline{1.027,148}|^3 + 3 \times \overline{1.027,148})^2 \times z + \&c$

$= \overline{1.027,148}|^3 \times 3 \times 1.055,0330 \times z + \&c$

$= \overline{1.027,148}|^3 + 3.165,099,0 \times z + \&c)$

$= 1.083,675,0 + 3.165,099,0 \times z + \&c$,

and x^4 ($= \overline{1.027,148 + z})^4 = \overline{1.027,148})^4 + 4 \times \overline{1.027,148})^3 \times z + \&c$

$= \overline{1.027,148})^4 + 4 \times 1.083,675,0 \times z + \&c$

$= \overline{1.027,148}|^4 + 4.334,700,0 \times z + \&c)$

$= 1.113,194,6 + 4.334,700,0 \times z + \&c.$

Therefore $11.24x$ will be ($= 11.24 \times \overline{1.027,148 + z} = 11.24 \times 1.027,148 + 11.24 \times z = 11.545,143,5 + 11.24z$,

and $21.76x^2$ will be ($= 21.76 \times \overline{1.055,033,0 + 2.054,296 \times z + \&c}$

$= 21.76 \times 1.055,033,0 + 21.76 \times 2.054,296 \times z + \&c)$

$= 22.957,518,0 \qquad + 44.701,480,9 \times z + \&c$,

and $11.54x^3$ will be ($= 11.54 \times \overline{1.083,675,0 + 3.165,099,0 \times z + \&c}$

$= 11.54 \times 1.083,675,0 + 11.54 \times 3.165,099,0 \times z + \&c)$

$= 12.505,609,5 \qquad + 36.525,242,4 \times z + \&c$;

and

and consequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be = the compound quantity

$$\left\{\begin{array}{rrl} 1.113{,}194{,}6 & + \ 4.334{,}700{,}0 & \times z + \&c \\ -\ 12.505{,}609{,}5 & -\ 36.525{,}242{,}4 & \times z - \&c \\ +\ 22.957{,}518{,}0 & +\ 44.701{,}480{,}9 & \times z + \&c \\ -\ 11.545{,}143{,}5 & -\ 11.24 & \times z \end{array}\right\}$$

$$= \left\{\begin{array}{rrl} 24.070{,}712{,}6 & +\ 49.036{,}180{,}9 & \times z + \&c \\ -24.050{,}753{,}0 & -\ 47.765{,}242{,}4 & \times z - \&c \end{array}\right\}$$

$$= 0.019{,}959{,}6 + 1.270{,}938{,}5 \times z$$

But the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ is $= 0.02$.

Therefore the compound quantity $0.019{,}959{,}6 + 1.270{,}938{,}5 \times z$ will also be $= 0.02$; and consequently $1.270{,}938{,}5 \times z$ will be $(= 0.020{,}000{,}0 - 0.019{,}959{,}6)$ $= 0.000{,}040{,}4$, and z will be $\left(= \frac{0.000{,}040{,}4}{1.270{,}938{,}5}\right) = 0.000{,}031{,}787$. Therefore x, or $1.027{,}148 + z$, will be $(= 1.027{,}148 + 0.000{,}031{,}787) = 1.027{,}179{,}787$; or the root of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ will be, very nearly, $= 1.027{,}179{,}787$.

Q. E. I.

Of this number 1.027,179,787, I believe the first nine figures, 1.027,179,78, to be exact: but we may be confident that, (if no mistakes have been made in the calculation,) at least the first seven figures, 1.027,179, will be exact,

exact, and consequently that the difference of this value of x from it's true value is less than 0.000,001, or one millionth part of 1, and, *à fortiori*, less than one millionth part of the true value of x. We have therefore now found (though not without a good deal of labour,) the value of this first, or least, root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ to a considerable degree of exactness.

Art. 34. But this least value of x will not enable us to solve Colonel Titus's Problem. For, if x is $=$ 1.027,179,787, we shall have x^2 ($= \overline{1.027,179,787}|^2$) $=$ 1.055,098,314,821,365,369, and consequently lx^2, or $16x^2$, will be $= 16 \times 1.055,098,314,821,365,369 =$ 16.881,573,037,141,845,904, which is less than 17, or m; whereas in the foregoing Solution of Colonel Titus's Problem, it is shewn that lx^2 is greater than m, or 17, and that y is $= \frac{lx^2 - m}{mx - l}$, or $\frac{16x^2 - 17}{17x - 16}$, which fraction, if $16x^2$ is less than 17, (as it is when x is $=$ 1.027,179,787,) is a quantity that cannot exist. We must therefore inquire, whether the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ has not another root, or, perhaps, two other roots, besides 1.027,179,787, and whether one or both the said roots may not be such as to enable us to solve the foregoing Problem.

Of

Of the Second and Third Roots of the Biquadratick Equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Art. 35. To find whether the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ has, or has not, any other root greater than 1.027,179,787, which has been already found, let us suppose that it has such a root, and inquire whether any false or impossible conclusion will follow from such a supposition. For, if no such false or impossible conclusion follows from it, but it should lead us to an equation that is evidently possible, the supposition itself must be concluded to be likewise true.

To avoid confusion in the reasonings that will be grounded on this supposition, let this supposed greater root of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ be denoted by the letter f, and the root 1.027,179,787 (which we have already found,) be denoted by the letter e.

Then we shall have $f^4 - 11.54f^3 + 21.76f^2 - 11.24f = 0.02$, and $e^4 - 11.54e^3 + 21.76e^2 - 11.24e$ also $= 0.2$, and consequently $f^4 - 11.54f^3 + 21.76f^2 - 11.24f = e^4 - 11.54e^3 + 21.76e^2 - 11.24e$. Therefore (adding $11.54f^3 + 11.24f$ to both sides,) we shall have $f^4 + 21.76f^2 = e^4 + 11.54f^3 - 11.54e^3 + 21.76e^2 + 11.24f - 11.24e$, and (subtracting $e^4 +$ $21.76e^2$

$21.76e^2$ from both fides,) $f^4 - e^4 + 21.76f^2 - 21.76e^2 = 11.54f^3 - 11.54e^3 + 11.24f - 11.24e$, or $f^4 - e^4 + 21.76 \times \overline{f^2 - e^2} = 11.54 \times \overline{f^3 - e^3} + 11.24 \times \overline{f - e}$. Therefore $\frac{f^4 - e^4}{f - e} + 21.76 \times \frac{\overline{f^2 - e^2}}{f - e}$ will be $= 11.54 \times \frac{f^3 - e^3}{f - e} + 11.24 \times \frac{f - e}{f - e}$, and confequently $f^3 + f^2e + fe^2 + e^3 + 21.76 \times \overline{f + e}$ will be $= 11.54 \times \overline{f^2 + fe + e^2} + 11.24$, or $f^3 + ef^2 + e^2f + e^3 + 21.76f + 21.76e$ will be $= 11.54f^2 + 11.54ef + 11.54e^2 + 11.24$; which is a cubick equation involving only one unknown quantity, to wit, f, the letter e denoting 1.027,179,787, or the root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, which has been already found. We muft therefore now inquire, whether this cubick equation $f^3 + ef^2 + e^2f + e^3 + 21.76f + 21.76e = 11.54f^2 + 11.54ef + 11.54e^2 + 11.24$ is a poffible equation.

Art. 36. In order to make this inquiry it will be neceffary to fubftitute, inftead of e, e^2, and e^3, in the terms of this equation their feveral values, to wit, 1.027,179,787, $\overline{1.027,179,787}|^2$ and $\overline{1.027,179,787}|^3$. But, as thefe values would lead us to very long numbers, and make thefe fubftitutions, and the fubfequent operations relating to this equation, extreamly laborious, I fhall fubftitute only the firft four figures of the value of e, to wit, 1.027, inftead of e on this occafion. And then, (fince it has been fhewn above, in art. 30, that $\overline{1.027}|^2$ is

$= 1.054,729$ and $\overline{1.027}|^3$ is $= 1.083,206,683$,) the ſaid cubick equation $f^3 + ef^2 + e^2f + e^3 + 21.76f + 21.76e = 11.54f^2 + 11.54ef + 11.54e^2 + 11.24$ will become $f^3 + 1.027 \times f^2 + 1.054,729 \times f + 1.083,206,683 + 21.76 \times f + 21.76 \times 1.027 = 11.54f^2 + 11.54 \times 1.027 \times f + 11.54 \times 1.054,729 + 11.24$, or $f^3 + 1.027f^2 + 1.054,729f + 1.083,206,683 + 21.76f + 22.347,52 = 11.54f^2 + 11.851,58f + 12.171,572,66 + 11.24$, or $f^3 + 1.027f^2 + 22.814,729f + 23.430,726,683 = 11.54f^2 + 11.851,58f + 23.411,572,66$; and conſequently (ſubtracting 23.411,572,66 from both ſides,) we ſhall have $f^3 + 1.027f^2 + 22.814,729f + 0.019,154,023 = 11.54f^2 + 11.851,58f$, and (ſubtracting $11.851,58f$ from both ſides) $f^3 + 1.027f^2 + 10.963,149f + 0.019,154,023 = 11.54f^2$, and (ſubtracting $1.027f^2$ from both ſides,) $f^3 + 10.963,149f + 0.019,154,023 = 10.513f^2$, and, laſtly, (ſubtracting $f^3 + 10.963,149f$ from both ſides,) $0.019,154,023 = 10.513f^2 - 10.963,149f - f^3$, or $10.513f^2 - 10.963,149f - f^3 = 0.019,154,023$. We muſt therefore now inquire, whether this laſt cubick equation $10.513f^2 - 10.963,149f - f^3 = 0.019,154,023$ is a poſſible equation, or whether it is poſſible for the ſingle quantity $10.513f^2$ to be greater than the binomial quantity $10.963,149f + f^3$, and to exceed it by the quantity 0.019,154,023.

Art. 37. Now, when f is $= 1.027$, this equation is impoſſible. For then f^2 is $= 1.054,729$, and f^3 is $= 1.083,206,683$, and $10.513f^2$ is $(= 10.513 \times 1.054,729)$

729) = 11.088,365,977, and $10.963,149f$ is (= $10.963,149 \times 1.027$) = 11.259,154,023, which is greater than 11.088,365,977. Therefore, in this case, the single term $10.963,149f$ will be greater than the single term $10.513f^2$; and therefore *à fortiòri* the binomial quantity $10.963,149f + f^3$ will be greater than the single term $10.513f^2$, and consequently cannot be subtracted from it, and therefore the equation $10.513f^2 - 10.963,149f - f^3 = 0.019,154,023$ (which supposes such subtraction to take place,) is impossible.

But, while f increases gradually from being equal to 1.027 to a greater magnitude, the single term $10.513f^2$ will receive greater increments than the binomial quantity $10.963,149f + f^3$, and will, first, become equal to it, and afterwards will exceed it. For the equation resulting from the supposition of it's becoming equal to it, to wit, the equation $10.513f^2 = 10.963,149f + f^3$ may easily be shewn to be a possible equation. For it will be a possible equation if the quadratick equation $10.513f = 10.963,149 + ff$ (arising from the division of all it's terms by f) is a possible equation: and this quadratick equation $10.513f = 10.963,149 + ff$, or $10.513f - ff = 10.963,149$, or $f \times \overline{10.513 - f}) = 10.963,149$, is possible, because 10.963,149 is less than the square of half 10.513, or the square of 5.256, which is greater than 25. Therefore the cubick equation $10.513f^2 = 10.963,149f + f^3$ is a possible equation.

And the magnitude of f at the instant at which $10.513f^2$ is exactly equal to $10.963,149f + f^3$, or

10.513f

$10.513f$ is $= 10.963,149 + ff$, may be found by resolving the quadratick equation $10.513f - ff = 10.963,149$; which may be done as follows.

Subtract both sides of this equation from the square of $\frac{10.513}{2}$, or 5.256, which is $= 27.625,536$; and we shall have $\overline{\frac{10.513}{2}}\Big|^2 - 10.513f + ff$ ($= 27.625,536 - 10.963,149$) $= 16.662,387$. Therefore $\frac{10.513}{2} - f$, or $5.256 - f$, will be $= \sqrt{16.662,387} = 4.081$, and consequently 5.256 will be $= 4.081 + f$, and f will be $= 5.256 - 4.081 = 1.175$. Also $f - 5.256$ will be $= 4.081$, and consequently f will be ($= 4.081 + 5.256$) $= 9.337$. Therefore, when f is $= 1.75$, the trinomial quantity $10.513f^2 - 10.963,149 - f^3$ will be $= 0$, and afterwards, when f is become $= 9.337$, the said trinomial quantity will be again $= 0$; and consequently, while f increases from 1.75 to 9.337, the trinomial quantity will have increased from 0 to some certain quantity, and have decreased from that quantity to 0; and therefore, if the said greatest quantity to which the said trinomial quantity will have increased is not less than $0.019,154,023$, the said trinomial quantity will, during it's said increase and decrease, or during the increase of f from 1.175 to 9.337, become twice equal to $0.019,154,023$, or the absolute term of the cubick equation $10.513f^2 - 10.963,149f - f^3 = 0.019,154,023$, to wit, once when f is a little greater than 1.175, and a second time when f is a little less than

9.337;

9.337; and consequently the said cubick equation will be a possible equation, and will have two roots, of which the lesser will be a little greater than 1.175, and the greater will be a little less than 9.337.

It only remains that we shew that the trinomial quantity $10.513f^2 - 10.963,149f - f^3$ will, in the course of it's increase and decrease while f increases from 1.175 to 9.337, become greater than 0.019,154,023. And this will appear by computing it's value at the instant that f becomes $= 2$. For, if f is $= 2$, we shall have $f^2 = 4$, and $f^3 = 8$, and $10.513 \times f^2$ $(= 10.513 \times 4) =$ 42.052, and $10.963,149f$ $(= 10.963,149 \times 2) =$ 21.926,298, and $10.963,149f + f^3$ $(= 21.926,298 + 8) = 29.926,298$, and $10.513f^2 - 10.963,149f - f^3$ $(= 42.052 - 29.926,298) = 12.125,702$; which is much greater than 0.019,154,023. It follows therefore that, during the increase of f from 1.175 to 2, the trinomial quantity $10.513f^2 - 10.963,149f - f^3$ will have increased from 0 to 12.125,702, and therefore must at some former instant of time, and when f was very little greater than 1.175, have been $= 0.019,154,023$. And consequently the cubick equation $10.513f^2 - 10.963,149f - f^3 = 0.019,154,023$ will be a possible equation, and will have two roots, of which the lesser will be a little greater than 1.175, and the greater will be a little less than 9.337. Q. E. I.

Art. 38. And hence it follows that the original biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ (from which the said cubick equation $10.513f^2$ —

$-$ 10.963,149$f - f^2$ = 0.019,154,023 was derived) will have two other roots besides the root 1.027,179, 787, found above, and that each of the said other roots will be greater than the said first root, and that the lesser of them will be a little greater than 1.175, and the greater of them will be a little less than 9.337. These roots we will now proceed to investigate.

The Investigation of the Middle Root of the Biquadratick Equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Art. 39. Since 1.175 is now known to be nearly equal to the middle root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, we will substitute it instead of x in the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, in order to discover how nearly the value of that compound quantity resulting from such substitution will approach to 0.02, or the absolute term of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Now, if x is = 1.175, we shall have x^2 $(= \overline{1.175}|^2)$ = 1.380,625, and x^3 $(= \overline{1.175}|^3)$ = 1.622,234,375, and x^4 $(= \overline{1.175}|^4)$ = 1.906,125,390,625. Therefore 11.24x will be (= 11.24 × 1.175) = 13.207,00, and 21.76x^2 will be (= 21.76 × 1.380,625) = 30.042, 400,00, and 11.54x^3 will be (= 11.54 × 1.622,234,375) =

$= 18.720,584,687,50$; and consequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $(= 1.906,125,390,625 - 18.720,584,687,50 + 30.042,400,00 - 13.207,00 = 31.948,525,390,625 - 31.927,584,687,50) = 0.020,940,703,125$. This number is very nearly equal to 0.02, or the absolute term of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$; and therefore 1.175 must be very nearly equal to this middle value of x in the said equation.

But as this value of the said compound quantity is greater than 0.02, or the absolute term of the equation, it may perhaps be thought that the number 1.175 is greater, instead of being less, (as it has been declared to be,) than the true value of this middle root of the said equation. And therefore, to ascertain this point, we will now substitute 1.176 instead of x in the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, in order to discover whether the said quantity will be increased, or diminished, by such small increase of x, and thence to determine whether 1.75 is greater, or less, than the true value of x.

Now, if x is $= 1.176$, we shall have x^2 $(= \overline{1.176}|^2)$ $= 1.382,976$, and x^3 $(= \overline{1.176}|^3) = 1.626,379,776$, and x^4 $(= \overline{1.176}|^4) = 1.912,622,616,576$. Therefore $11.24x$ will be $(= 11.24 \times 1.176) = 13.218,24$, and $21.76x^2$ will be $(= 21.76 \times 1.382,976) = 30.093,547,76$, and $11.54x^3$ will be $(= 11.54 \times 1.626,379,776) = 18.768,422,615,04$; and consequently the whole com-

pound

pound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $(= 1.912,622,616,576 - 18.768,422,615,04 + 30.093,547,76 - 13.218,24 = 32.006,170,376,576 - 31.986,662,615,04) = 0.019,507,721,536$; which is less than 0.020,940,703,125, or the former value of the said compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, when x was $= 1.175$, and also less than 0.02, or the absolute term of the equation under consideration. Therefore, while x increases from 1.175 to 1.176, the said compound quantity will decrease from 0.020,940,703,125 (which is greater than the absolute term 0.02,) to 0.019,507,721,536 (which is less than the said absolute term,) and therefore will, at some intermediate instant of time, or when x was of some intermediate magnitude between 1.175 and 1.176, have been equal to the said absolute term; or, in other words, the true value of this root of the said equation will be of an intermediate magnitude between 1.175 and 1.176. Q. E. D.

Art. 40. We are now in possession of two values of the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, to wit, the numbers 0.020,940,703,125 and 0.019,507,721,536, corresponding to the two very near values of x, 1.175 and 1.176, between which the true value of x in the present biquadratick equation lies. We may therefore now obtain a much nearer value of x than either 1.175 or 1.176 by proceeding according to the differential method mentioned above in the Scholium in page 97. For, since 1.175, and the true value of x in the present equation, and 1.176, are three quantities, or values of x, very nearly equal to each other, and the

three values of the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ correſponding to the ſaid three contiguous quantities, or values of x, (or reſulting from the ſubſtitution of them in the ſaid compound quantity,) are 0.020,940,703,125, 0.020,000,000, and 0.019,507,721,536, we may ſuppoſe that the difference of the firſt and ſecond values of x will be to the difference of the firſt and third values of x in nearly the ſame proportion as the difference of the values of the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ correſponding to the firſt and ſecond values of x to the difference of the values of the ſaid compound quantity correſponding to the firſt and third values of x; that is, that $x - 1.175$ will be to $1.176 - 1.175$ in nearly the ſame proportion as $0.020,940,703,125 - 0.020,000,000$ is to $0.020,940,703,125 - 0.019,507,721,536$, or that $x - 1.175$ will be to 0.001 in nearly the ſame proportion as 0.000,940,703,125 is to 0.001,432,981,589; whence it follows that $x - 1.175$ will be nearly $=$

$$\frac{0.001 \times 0.000,940,703,125}{0.001,432,981,589} = \frac{0.000,000,940,703,125}{0.001,432,981,589} =$$

0.000,656,46, and conſequently that x will be ($=$ 0.000,656,46 + 1.175) = 1.175,656,46. Therefore this middle root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ will be $= 1.175,656,46$.

Q. E. I.

If no miſtake has been made in the foregoing calculations, this value of x will be exact in, at leaſt, the firſt ſix figures 1.175,65, and, probably, in the firſt ſeven figures 1.175,656.

The

The Application of this Second Value of x *in the Biquadratick Equation* $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, *or the Middle Root of the said Equation, to the Solution of Colonel Titus's Problem.*

Art. 41. We will now apply this value of x to the solution of Colonel Titus's Problem.

Since x is $= 1.175,656$, and b, or the second of the unknown numbers a, b, and c that are required to be found, is supposed to be $= xa$, we shall have $b = 1.175,656 \times a$.

Also we shall have x^2 $(= \overline{1.175,656}|^2) = 1.382,167,030,336$, and consequently lx^2, or $16x^2$, $(= 16 \times 1.382,167,030,336) = 22.114,672,485,376$, and $lx^2 - m$ $(= 16x^2 - 17 = 22.114,672,485.376 - 17) = 5.114,672,485,376$, and $mx - l$ $(= 17x - 16 = 17 \times 1.175,656 - 16 = 19.986,152 - 16) = 3.986,152$, and consequently $\frac{lx^2 - m}{mx - l} = \frac{5.114,672,485,376}{3.986,152} = 1.283,110,2$. Therefore y (which is $= \frac{lx^2 - m}{mx - l}$, or $\frac{16x^2 - 17}{17x - 16}$) will be $= 1.283,110,2$, or (neglecting the last figure) $1.283,110$; and c (which is $= ya$) will be $= 1.283,110 \times a$.

Therefore bc will be $= 1.175,656 \times a \times 1.283,110 \times a$ $(= 1.175,656 \times 1.283,110 \times aa = 1.508,495,970,160 \times aa$, and $aa + bc$ will be $= aa + 1.508,495,970,160 \times aa = 2.508,495,970,160 \times aa$.

But $aa + bc$ is $= 16$.

Therefore

Therefore $2.508,495,970,160 \times aa$ will alſo be $= 16$; and conſequently the ſquare-root of $2.508,495,970,160 \times aa$ will be $=$ the ſquare-root of 16, or to 4; that is, $1.583,823 \times a$ will be $= 4$; and conſequently a will be

$$= \frac{4}{1.583,823} = 2.525,534.$$

This value of a is exact in the firſt five figures 2.525,5, and ſomewhat too great in the ſixth figure 3, which ought to be a 1, the more accurate value of it, according to Dr. Wallis's computation of it, being 2.525,513,986, 744,158.

Art. 42. Having thus found a to be $= 2.525,534$, we ſhall have b $(= xa = 1.175,656 \times a = 1.175,656 \times 2.525,534) = 2.969,159$, and c $(= ya = 1,283,110 \times a = 1.283,110 \times 2.525,534) = 3.240,537$; of which two numbers the former, to wit, 2.969,159, is exact in the firſt ſix figures 2.969,15, and the latter, to wit, 3.240,537, is exact in the firſt five figures 3.240,5, the more accurate values of b and c being, according to Dr. Wallis's computation of them, 2.969,152,7[illegible]8,619, 848 and 3.240,580,681,617,174. Therefore the three unknown numbers a, b, and c, which Colonel Titus's Problem required to be found, are now diſcovered to be 2.525,5 &c, 2.969,15 &c, and 3.240,5 &c. Q. E. I.

Art. 43. We have now found the firſt ſet of values of the unknown numbers a, b, and c, that will anſwer the conditions of Colonel Titus's Problem, which are 2.525,5 &c, 2.969,15 &c, and 3.240,5 &c. But there is alſo a ſecond ſet of numbers that will anſwer the conditions of the

the Problem. And theſe may be diſcovered by means of the third, or greateſt, root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, which we have already diſcovered to be nearly equal to, and ſomewhat leſs than, 9.337. This root we will therefore now proceed to inveſtigate to a greater degree of exactneſs.

The Inveſtigation of the Greateſt Root of the Biquadratick Equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Art. 44. In the firſt place we will ſubſtitute 9.337 inſtead of x in the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, in order to diſcover how near the value of the ſaid compound quantity reſulting from this ſubſtitution will approach to 0.02, or the abſolute term of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Now, if x is $= 9.337$, we ſhall have x^2 $(= \overline{9.337}|^2)$ $= 87.179,569$, and x^3 $(= \overline{9.337}|^3) = 813.995,635,753$, and x^4 $(= \overline{9.337}|^4) = 7600.277,251,025,761$, and $11.24x$ $(= 11.24 \times 9.337) = 104.947,88$, and $21.76x^2$ $(= 21.76 \times 87.179,569) = 1897.027,421,44$, and $11.54x^3$ $(= 11.54 \times 813.995,635,753) = 9393.509,636,589,62$. Therefore the whole compound quantity

$x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be (= 7600.277,251,025,761 — 9393.509,636,589,62 + 1897.027,421,44 — 104.947,88) = 9497.304,672,465,761 — 9498.457,516,589,62. This quantity is an impoſſible quantity, becauſe the latter quantity 9498.457,516,589,62 is greater than the former quantity 9497.304,672,465,761, from which it is to be ſubtracted. This unexpected reſult is owing to our having, in art. 36, ſubſtituted 1.027 inſtead of 1.027,179,787 for e in the terms of the equation $f^3 + ef^2 + e^2f + e^3 + 21.76f + 21.76e = 11.54f^2 + 11.54ef + 11.54e^2 + 11.24$; which was done for the ſake of abridging the calculations neceſſary to thoſe ſubſtitutions, but which made the final equation $10.513f^2 - 10.963,149f - f^3 = 0.019,154,023$ a little different from what it ſhould have been, and thereby gave us the two values of f derived from the ſuppoſition that the abſolute term of that equation was equal to 0, (and which were found in art. 37, by reſolving the quadratick equation $10.513f - ff = 10.963,149$, to be 1.175 and 9.337) a little different from what they ſhould have been, the former value of f in that equation, to wit, 1.175, being a little greater than it ſhould be, and the latter value of it, to wit, 9.337, being a little leſs than it ſhould be. We will therefore now ſuppoſe x, or the greateſt root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, of which we are now in ſearch, to be nearly equal to 9.339, and will ſubſtitute 9.339 inſtead of x in the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, in order to diſcover how near the reſult of ſuch ſubſtitution will approach to the abſolute term 0.02 of the ſaid biquadratick equation.

Art. 45.

Art. 45. Now, if x is $= 9.339$, we shall have xx ($= \overline{9.339}|^2$) $= 87.216,921$, and x^3 ($= \overline{9.339}|^3$) $= 814.518,825,219$, and x^4 ($= \overline{9.339}^4$) $= 7606.791,308,720,241$, and $11.24x$ ($= 11.24 \times 9.339$) $= 104.970,36$, and $21.76x^2$ ($= 21.76 \times 87.216,921$) $= 1897.840,200,96$, and $11.54x^3$ ($= 11.54 \times 814.518,825,219$) $= 9399.547,243,027,26$, and $x^4 + 21.76x^2$ ($= 7606.791,308,720,241 + 1897.840,200,96$) $= 9504.631,509,680,241$, and $11.54x^3 + 11.24x$ ($= 9399.547,243,027,26 + 104.970,36$) $= 9504.517,603,027,26$. Therefore the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $= 9504.631,509,680,241 - 9504.517,603,027,26 = 0.113,906,652,981$; which is more than five times 0.02, or the absolute term of the proposed equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. We therefore now know with certainty that, while x increases from 9.337 to 9.339, the binomial quantity $x^4 + 21.76x^2$ (which, when x is equal to 9.337, is less than the binomial quantity $11.54x^3 + 11.24x$) will have increased so as, first, to have become equal to the binomial quantity $11.54x^3 + 11.24x$, and afterwards to have become greater than the said binomial quantity by the difference 0.113,906,652,981, which is greater than five times the absolute term 0.02. And therefore it is certain that during the said increase of x from 9.337 to 9.339, there must have been some one instant of time at which the excess of the binomial quantity $x^4 + 21.76x^2$ above the binomial quantity $11.54x^3 + 11.24x$ was equal to 0.02, or the absolute term of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$; or, in other

other words, there muſt be ſome value of x greater than 9.337, but leſs than 9.339, that will make the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ (which is the exceſs of the binomial quantity $x^4 + 21.76x^2$ above the binomial quantity $11.54x^3 + 11.24x$) exactly equal to 0.02, or will be a root of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. This root we will therefore now proceed to inveſtigate more exactly by Mr. Raphſon's method of approximation, and for that purpoſe will ſuppoſe x to be equal to $9.339 - z$, and will ſubſtitute $9.339 - z$ inſtead of it in the terms of the ſaid equation.

Art. 46. Now, if x is $= 9.339 - z$, we ſhall have xx $(= \overline{9.339 - z}|^2 = \overline{9.339}|^2 - 2 \times 9.339 \times z +$ &c $= \overline{9.339}|^2 - 18.678 \times z +$ &c$) = 87.216,921, - 18.678 \times z +$ &c,

and x^3 $(= \overline{9.339 - z}|^3 = \overline{9.339}|^3 - 3 \times \overline{9.339}|^2 \times z +$ &c

$= \overline{9.339}|^3 - 3 \times 87.216,921 \times z +$ &c

$= \overline{9.339}|^3 - 261.650,763 \times z +$ &c$)$

$= 814.518,825,219 - 261.650,763 \times z +$ &c,

and x^4 $(= \overline{9.339 - z}|^4 = \overline{9.339}|^4 - 4 \times \overline{9.339}|^3 \times z +$ &c

$= \overline{9.339}|^4 - 4 \times 814.518,825,219 \times z +$ &c

$= \overline{9.339}|^4 - 3258.075,300,876 \times z +$ &c$)$

$= 7606.791,308,720,241 - 3258.075,300,876 \times z +$ &c.

Therefore

Therefore $11.24x$ will be $(= 11.24 \times \overline{9.339 - z} = 11.24 \times 9.339 - 11.24 \times z) = 104.970,36 - 11.24z$; and $21.76x^2$ will be $(= 21.76 \times \overline{87.216,921 - 18.678 \times z + \&c} = 21.76 \times 87.216,921 - 21.76 \times 18.678 \times z + \&c) = 1897.840,200.96 - 406.433,28 \times z + \&c$, and $11.54x^3$ will be $(= 11.54 \times \overline{814.518,825,219 - 261.650,763 \times z + \&c} = 11.54 \times 814.518,825,219 - 11.54 \times 261.650,763 \times z + \&c) = 9399.547,243,027,26 - 3019.449,805,02 \times z + \&c$. And consequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be equal to the compound quantity

$$\left\{\begin{array}{llll} & 7606.791,308,720,241 & - 3258.075,300,876 & \times z + \&c \\ - & 9399.547,243,027,26 & + 3019.449,805,02 & \times z - \&c \\ + & 1897.840,200,96 & - 406.433,28 & \times z + \&c \\ - & 104.970,36 & + 11.24 & \times z \end{array}\right\}$$

$$= \left\{\begin{array}{lll} 9504.631,509,680,241 & - 3664.508,580,876 & \times z + \&c \\ -9504.517,603,027,26 & + 3030.689,805,02 & \times z - \&c \end{array}\right\}$$

$$= 0.113,906,652,981 \quad - \quad 633.818,775,856 \times z \ \&c.$$

But the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ is $= 0.02$.

Therefore the compound quantity $0.113,906,652,981 - 633.818,775,856 \times z$ will also be $= 0.02$. And consequently (adding $633.818,775,856 \times z$ to both sides,) we shall have $0.113,906,652,981 = 0.02 +$

633.818,

$633.818,775,856 \times z$, and (subtracting 0.02 from both sides,) $0.093,906,652,981 = 633.818,775,856 \times z$, and (dividing both sides by 633.818,775,856) we shall have $z\ (= \frac{0.093,906,652,981}{633.818,775,856}) = 0.000,148,1$. Therefore x, or $9.339 - z$, will be $(= 9.339,000,0 - 0.000,148,1) = 9.338,851,9$; that is, the third and greatest root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ will be $= 9.338,851,9$.

Q. E. I.

Art. 47. We will now apply this greatest value of x to the solution of Colonel Titus's Problem.

Since x is $= 9.338,851,9$, and b, or the second of the three unknown numbers a, b, and c that are required to be found, is $= xa$, we shall have $b = 9.338,851,9 \times a$.

Also we shall have $x^2\ (= \overline{9.338,851,9}|^2) = 87.214,154,810,133,61$, and consequently $lx^2\ (= 16x^2 = 16 \times 87.214,154,810,133,61) = 1395.426,476,962,137,76$, and $lx^2 - m\ (= 16x^2 - 17 = 1395.426,476,962,137,76 - 17) = 1378.426,476,962,137,76$, and $mx\ (= 17x = 17 \times 9.338,851,9) = 158.760,482,3$, and $mx - l\ (= 17x - 16 = 158.760,482,3 - 16) = 142.760,482,3$, and consequently $\frac{lx^2 - m}{mx - l} = \frac{1378.426,476,962,137,76}{142.760,482,3} = 9.655,518,4$. Therefore y

T (which

(which is $= \frac{lx^2 - m}{mx - l}$, or $\frac{16x^2 - 17}{17x - 16}$) will be $= 9.655,518,4$, and c (which is $= ya$) will be $= 9.655,518,4 \times a$.

Therefore bc will be $(= 9.338,851,9 \times a \times 9.655,518,4 \times a = 9.338,851,9 \times 9.655,518,4 \times aa) = 90.171,456,355,324,96 \times aa$, and $aa + bc$ will be $= aa + 90.171,456,355,324,96 \times aa = 91.171,456,355,324,96 \times aa$.

But $aa + bc$ is $= 16$.

Therefore $91.171,456,355,324,96 \times aa$ will be $= 16$; and consequently the square-root of $91.171,456,355,324,96 \times aa$ will be $=$ the square-root of 16, or to 4; that is, $9.548,374,5 \times a$ will be $= 4$; and consequently

$$a \text{ will be } = \frac{4}{9.548,374,5} = 0.418,919,47.$$

This value of a is exact in all it's eight figures, in which it agrees with the value of it computed by Dr. Wallis to nine places of figures, which is 0.418,919,470. This is a very considerable degree of exactness.

Art. 48. Having thus found a to be $= 0.418,919,47$, we shall have b $(= xa = 9.338,851,9 \times a = 9.338,851,9 \times 0.418,919,47) = 3.912,226,888$, and c $(= ya = 9.655,518,4 \times a = 9.655,518,4 \times 0.418,919,47) = 4.044,884,650$; of which two numbers the former, to wit, 3.912,226,888, is exact in the first eight figures,

3.912,

3.912,226,8, and the latter, to wit, 4.044,884,650, is alſo exact in the firſt eight figures, 4.044,884,6; the more accurate values of *b* and *c*, according to Dr. Wallis's computation of them, being 3.912,226,866 and 4.044,884,670. Therefore the three unknown numbers *a*, *b*, and *c*, which Colonel Titus's Problem required to be found, are now diſcovered to be 0.418,919,47 &c, 3.912,226,8, &c, and 4.044,884,6, &c. Q. E. I.

This Problem is therefore now compleatly ſolved by Mr. Frend's method of proceeding, as it had been before by that of Dr. Wallis: and Mr. Frend's Solution has this advantage over that of Dr. Wallis, that it ſaves us the trouble of thoſe very tedious and perplexing Algebräick multiplications and diviſions which were neceſſary in Dr. Wallis's Solution, and which it is very difficult to perform without making ſome ſlip, or error, either in the ſigns + and — that are to be prefixed to the terms, or in the values of them.

OBSERVATIONS

ON

MR. RAPHSON's METHOD OF RESOLVING AFFECTED EQUATIONS OF ALL DEGREES

BY

APPROXIMATION.

Observations on Mr. Raphson's *Method of Resolving Affected Equations of all Degrees by Approximation.*

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[REPRINTED FROM A FORMER PUBLICATION.]

Article 1. IN the foregoing Tract I have given a pretty full explanation of Monsieur de Lagny's Method of Extracting the Roots of Numbers by Approximation, and I have likewise mentioned Mr. Raphson's more simple and easy, though less exact, method of performing the same thing. But both these methods may be applied to the resolution of all sorts of equations, those which are called *affected* equations *, or in which the unknown quantity occurs in more than one term,

* This expression of *affected equations* seems to require some further explanation. It was introduced by the celebrated Vieta, the great improver of modern Algebra. He has many expressions peculiar to himself, and which have not been adopted by subsequent Algebräists. Amongst these are the following ones: —He calls a set of quantities that are in continual geometrical proportion,

term, as well as thoſe which are called *pure* equations or in which the unknown quantity occurs in only one term, and which are reſolved by the mere extraction of the roots of given numbers. And in all affected equations beyond

proportion, (ſuch as the quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c,) a ſet of *ſcalar* quantities, or *magnitudines ſcalares*; and, when there are ſeveral of theſe *ſcalar* quantities connected with each other by the ſigns + and −, or by Addition and Subtraction, (as in the compound quantity $x^5 + ax^4 - b^2x^3$,) he calls the higheſt quantity, ~~or that~~ which is fartheſt in the ſcale of quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c, (to wit, the quantity x^5 in the ſaid compound quantity $x^5 + ax^4 - b^2x^3$,) *the power* of the fundamental quantity x, or of the ſecond term in the ſaid ſcale; and he calls the lower ſcalar quantities, which are involved in the ſecond and third terms of the ſaid compound quantity $x^5 + ax^4 - b^2x^3$, to wit, the quantities x^4 and x^3, (or, in our preſent language, the inferiour powers of x,) ſcalar quantities of a *parodic* degree to x^5, or *the power* of the fundamental quantity x. This word *parodic* I take to be derived (though Vieta does not tell us ſo,) from the Greek words παρα and ὁδὸς, which ſignify *near* and *a way*, or *road*, becauſe theſe inferiour ſcalar quantities, x^3 and x^4, lie *in the way* as you paſs along in the ſcale of the aforeſaid quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c, from 1 to x^5, which he calls *the power* of x in the ſaid compound quantity $x^5 + ax^4 - b^2x^3$. Theſe inferiour ſcalar quantities x^3 and x^4 are therefore *parodic*, or *ſituated in the way to*, or are *leading to*, the ſaid *power*, or higher ſcalar quantity, x^5. He then proceeds to define *a pure power* and *an affected power*, and tells us, that *a pure power* is a ſcalar quantity that is not affected with, or mixed with, any *parodic*, or *inferiour* ſcalar quantity, and that *an affected power* is a ſcalar quantity

beyond biquadraticks, or thofe of the fourth power, thefe methods of approximation are the only methods that can be taken for difcovering their roots, or the values of the unknown quantities contained in them. And even in cubick

quantity that is mixed, or connected by Addition, or Subtraction, with one, or more, *inferiour*, or *parodic*, fcalar quantities, combined with co-efficients that raife them to the fame dimenfion as the *power* itfelf, or make them *homogeneous* to it, and confequently capable of being added to it, or fubtracted from it. Thus x^5 alone is *a pure power* of x, namely, its fifth power; and $x^5 + ax^4 - b^2x^3$ is *an affected power* of x, namely, it's fifth power *affected by*, or *connected with*, the two *parodic*, or *inferiour* fcalar quantities, x^3 and x^4, which are multiplied into bb and a, in order to make them *homogeneous to*, or *of the fame dimenfion with*, x^5 itfelf, and confequently capable of being added to it, or fubtracted from it. And he feems to have ufed the word *affected* in fpeaking of fuch a compound quantity as $x^5 + ax^4 - b^2x^3$, becaufe the magnitude of the higheft power of x, to wit, x^5, was changed, or altered, or made greater, or lefs, than it would otherwife be, by the addition of the parodic quantity ax^4, and the fubtraction of the parodic quantity b^2x^3; which increafe, or diminution, or change, in the principal quantity x^5 he feemed to think might be well expreffed by faying it was *affected* by it's connection with the faid parodic quantities. There are fome expreffions in his book that fatisfy me that this was the idea he annexed to the word *affected*. See Schooten's edition of Vieta's Works, publifhed at Leyden in Holland, in the year 1646, pages 3 and 4.

This, then, being the meaning of the expreffions *a pure power* and *an affected power*, the meaning of the correfponding expreffions

cubick and biquadratick equations, though particular methods have been invented by mathematicians, for the accurate resolution of most of the cases of these equations, (to wit, the rules called Cardan's Rules for the resolution of most cases of cubick equations, and the rules invented by Lewis Ferrari of Bologna in Italy, about the year 1545, and explained at large in Bombelli's Algebra, in the year 1579, and those afterwards invented by M. Des Cartes, and published in his Geometry in the year 1637, for the resolution of biquadratick equations, by the mediation of cubick equations,) it will be found that these methods of approximation will, for the most part,

pressions of *a pure equation* and *an affected equation* follows from it of course; *a pure equation* signifying an equation in which a pure power of an unknown quantity is declared to be equal to some known quantity; such as the equation $x^5 = 79$; and *an affected equation* signifying an equation in which a power of an unknown quantity affected by, or connected, either by Addition or Subtraction, with, some inferiour powers of the same unknown quantity, (multiplied into proper co-efficients in order to make them *homogeneous* to the said highest power of the said unknown quantity,) is declared to be equal to some known quantity; such as the equation $x^5 + ax^4 - b^2x^3 = 79$. This I take to be the original meaning of the expression *an affected equation*. But, as the language of Vieta has not been adopted by subsequent writers of Algebra, I should think it would be more convenient to call them by some other name. And, perhaps, those of *binomial, trinomial, quadrinomial, quinquinomial*, and, in general, that of *multinomial* equations, would be as convenient as any. Thus, $xx + ax = rr$, and $x^3 + ax^2 = r^3$, and $x^3 + a^2x = r^3$, and

part, enable us to find the values of their roots to any propofed degree of exactnefs, with lefs trouble than the particular and accurate methods above-mentioned, which have been invented for that purpofe. So that thefe methods of refolving equations by approximation ought to be confidered as of the higheft utility, and as being abfolutely neceffary to the completion of the Doctrine of the Refolution of Algebräick Equations, which is the moft important branch of the Science of Algebra.

Art. 2. But it is not fo eafy to determine, which of thefe two methods of approximation, Mr. Raphfon's, or Mr. de Lagny's, deferves to be preferred to the other on

and $x^4 + a^3x = r^4$, and $x^4 + ax^3 = r^4$, might all be called *binomial* equations, becaufe they would be equations in which a *binomial* quantity, or quantity confifting of two terms that involved the unknown quantity x, is declared to be equal to a known quantity; and, for a like reafon, the equations $x^3 + ax^2 + b^2x = r^3$, and $x^4 - ax^3 + b^2x^2 = r^4$, and $x^4 - ax^3 + b^3x = r^4$, and $x^5 + ax^4 + b^2x^3 = r^5$, and $x^5 + ax^4 - b^2x^3 = r^5$, and $x^5 + b^2x^3 + c^4x = r^5$, might be called *trinomial* equations. And the like names might be given to equations of a greater number of terms. Dr. Hutton, I obferve, in his excellent new Mathematical and Philofophical Dictionary, juft now publifhed (Feb. 2, 1795,) calls them *compound* equations; which is likewife a very proper name for them, and lefs obfcure than that of *affected* equations. And Mr. Kerfey, in his excellent Treatife of Algebra, publifhed in the year 1673, likewife calls them *compound* equations. But I fhould think the former appellation of *binomial, trinomial, quadrinomial, quinquinomial,* and, in general, *multinomial* equations would be rather more defcriptive of their nature.

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these occasions. Mr. Raphson's is certainly much simpler than the other, because it proceeds by considering the new, or transformed, equation, (resulting from the substitution of $a + z$, or $a - z$, instead of x, in the original equation,) as being only a simple equation, and resolving it accordingly, or by the mere operation of Division; whereas, in Mr. de Lagny's method, the said new, or transformed, equation is considered as a quadratick equation, and resolved accordingly; which, when a (or the first near value of the root, that is supposed to be already known,) is a number consisting of five, or six, figures, produces a great deal of labour, and often a great deal of perplexity. I am therefore inclined to give the preference to Mr. Raphson's method in resolving all affected equations, more especially when the number a consists of more than two figures: but it must be confessed that the celebrated Dr. Halley (who had much experience, and was an excellent judge of these matters,) was of a different opinion, and gave the preference to Mr. de Lagny's method, which he has therefore taken the pains to explain in a better manner than had been done by Mr. de Lagny himself, and likewise to illustrate by examples, in his Tract in the Philosophical Transactions, Number 210, intitled, "*A New, Exact, and Easy, Method of finding the* "*Roots of any Equations Generally, and that without any* "*previous Reduction,*" which was published in the year 1694. On the other hand we may observe, that Mr. Raphson always continued to give his own method the preference, after the publication of the tracts of Mr. de Lagny and Dr. Halley upon the subject, as well as before their publication, when he tells us he had himself had the thought

thought of adopting the principle which was afterwards followed by Mr. de Lagny and Dr. Halley, of treating the transformed equation as a quadratick equation, but had deliberately rejected it on account of the greater ease and simplicity of the other method, in which the said transformed equation is considered and treated as a simple equation. And Sir Isaac Newton, in his method of resolving equations by approximation, (which differs very little from Mr. Raphson's,) seems also to prefer Mr. Raphson's practice, of treating the transformed equation as a more simple equation, to that of Mr. de Lagny and Dr. Halley, of treating the said equation as a quadratick equation. I therefore cannot but recommend it to all young Algebräists to study Mr. Raphson's excellent Treatise on this subject, intitled, *Analysis Æquationum Universalis*, with great attention, and to endeavour to make themselves masters of it, by going carefully through all the examples given in it, and performing all the arithmetical operations contained in them. And I will venture to say that they will thereby acquire more useful knowledge in Algebra, towards the business of resolving affected, or compound, or multinomial, equations, than by reading all that has been written by Harriot and Des Cartes, and his learned Commentator Van Schooten, and all his other Commentators, and their numerous followers, on the boasted doctrine of the Generation of Equations one from another, by supposing $x - a$ to be $= 0$, and $x - b$ to be $= 0$, and $x - c$ to be $= 0$, and $x + d$ to be $= 0$, and $x + e$ to be $= 0$, and so on; and then multiplying the binomial quantities $x - a$, $x - b$, $x - c$, $x + d$, $x + e$, &c, into each other, and like-

wiſe all the abſtruſe and intricate matter that has been delivered by Sir Iſaac Newton, and Mr. Graveſende and Mr. Mac Laurin, and other learned Algebräiſts of modern times, on the invention of Diviſors, which is grounded on that doctrine of the Generation of Equations from each other.

Art. 3. Yet in reading this excellent Treatiſe of Mr. Raphſon, which I ſo much recommend, there will now and then occur ſome difficulties which are not inherent in the ſubject itſelf, but which might have been avoided, if Mr. Raphſon had not unfortunately adopted the perplexing doctrines of modern writers of Algebra, about negative quantities and negative roots of equations. The quantities called *negative* are ſuch as it is impoſſible to form any clear idea of, being defined, by Sir Iſaac Newton and other Algebräiſts *, to be ſuch quantities as are *leſs than nothing*, or as *ariſe from the ſubtraction of a greater quantity from a leſſer*, which is an operation evidently im-

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* Quantitates vel *Affirmativæ* ſunt, ſeu majores Nihilo, vel *Negativæ*, ſeu Nihilo minores.—Newton's *Arithmetica Univerſalis*, page 3.

When a greater quantity is taken from a leſſer of the ſame kind, the remainder becomes of the oppoſite kind.—*Mac Laurin's Algebra*, page 5.

An affirmative quantity is a quantity greater than nothing, and is known by this ſign, +; a negative quantity is a quantity leſs than nothing, and is known by this ſign, −.—*Saunderſon's Algebra*, Vol. I. page 50, Article 2.

poſſible

poſſible to be performed: and, as to the *negative roots* of an equation, they are in truth the real and poſitive roots of another equation conſiſting of the ſame terms as the firſt equation, but with different ſigns + and − prefixed to ſome of them; ſo that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, and ſuppoſe the propoſed, or firſt, equation to have not only it's own proper roots (which they call it's *affirmative*, or *poſitive*, roots,) but to have likewiſe the roots of a different equation, which they call it's *negative* roots. Thus, for example, they would ſay, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the poſitive, or affirmative, root, + 16, and the negative root, — 20. But this latter number, 20, is, in truth, the root of a different equation, to wit, of the equation $xx - 4x = 320$. So that this kind of abſurd and fantaſtick language only tends to the confounding together the two different equations $xx + 4x = 320$, and $xx - 4x = 320$, and conſidering them as if they were one and the ſame equation. Now this perplexing language is unfortunately uſed by Mr. Raphſon in this valuable Treatiſe, and tends to throw an air of myſtery and obſcurity upon ſome of the Problems ſolved in it, from which they would otherwiſe have been intirely free. As a proof of the truth of this obſervation, I ſhall here inſert one of the ſaid Problems, the ſolution of which is by this means rendered ſo obſcure, that I had a good deal of trouble to find out the meaning of it; though, if this language had been avoided, and the proper and natural language, belonging to the conditions of the Problem, had been uſed in it's ſtead, there could not have been

been the leaſt difficulty in underſtanding it. This Problem is the 24th, in page 32 of the 2d edition of the book, and is, *verbatìm et litteratìm*, as follows:

PROBLEMA XXIV.

ÆQUATIONUM QUINTÆ POTESTATIS ADFECTARUM SOLUTIO.

Proponatur $-aaaaa + 7aaaa - 20aaa + 155aa = 10{,}000.$

Hoc eſt, $-aaaaa + baaaa - caaa + daa = f.$

$$\text{Theor. } x = \frac{f + ggggg + cggg - bgggg - dgg}{4bggg + 2dg - 5gggg - 3cgg}$$

Sit $g = -5$

$$f + ggggg + cggg - bgggg - dgg = -3875$$

$$4bggg + 2dg - 5gggg - 3cgg = -9675)\ (-3875{,}0)\ +{,}4 = x$$

$$\begin{array}{r} -5, \\ +{,}4 \\ \hline g = -4{,}6 \end{array}$$

$$f + ggggg + cggg - bgggg - dgg = -420{,}36896$$

$$4bggg + 2dg - 5gggg - 3cgg = -7659{,}736)\ -420{,}36896\ (+{,}055 = x$$

$$\begin{array}{r} -4{,}6 \\ +\ {,}055 \\ \hline g = -4{,}545 \end{array}$$

$f +$

$$f + ggggg + cggg - bgggg - dgg = -5{,}9603594654656\,25$$
$$4bggg + 2dg - 5gggg - 3cgg = -7410{,}748) - 5{,}9603594$$
$$(+ {,}00080428 = x$$

$$- 4{,}545$$
$$+ \ {,}000{,}80428$$

$$a = - 4{,}544{,}195{,}72 \ *$$

In this Problem the letter *a* is used for the unknown quantity, or root of the equation, which is usually denoted by the letter *x*; and the letter *g* is used for the first near value of the root of the equation, which in the two foregoing Tracts has been denoted by the letter *a*; and the letter *x* is used for the difference between *g*, the first near value of the root of the equation, and *a*, it's true value, which difference has been denoted in the two foregoing Tracts by the letter *z*. So that, if we express the enun-

* To this solution I have, in my copy of Mr. Raphson's Tract, subjoined the following Note:

Numerus 4.544,195,72 est radix æquationis $a^5 + 7a^4 + 20a^3 + 155a^2 = 10{,}000$; quod hic obscurè innuitur sub specie radicis negativæ æquationis $-a^5 + 7a^4 - 20a^3 + 155a^2 = 10{,}000$. Omnes ferè difficultates quibus permulti cultioris ingenii viri ab Algebrâ discendâ et excolendâ deterrentur, ex hisce radicibus negativis et aliis quantitatibus negativis, seu (ut hodierni Algebræ scriptores absurdè loquuntur,) *Nihilo minoribus*, ortum habent.

ciation of the foregoing Problem in the notation that has been used in the two foregoing Tracts, it will be as follows:

$$\text{Proponatur} - xxxxx + 7xxxx - 20xxx + 155xx = 10{,}000,$$

$$\text{Sive} - x^5 + 7x^4 - 20x^3 + 155xx = 10{,}000,$$

$$\text{Hoc eſt,} - xxxxx + bxxxx - cxxx + dxx = f,$$

$$\text{Sive} - x^5 + bx^4 - cx^3 + dx^2 = f.$$

$$\text{Theor. } z = \frac{f + aaaaa + caaa - baaaa - daa}{4baaa + 2da - 5aaaa - 3caa},$$

$$\text{or } z = \frac{f + a^5 + ca^3 - ba^4 - da^2}{4ba^3 + 2da - 5a^4 - 3ca^2}.$$

Art. 4. Here, then, the equation propoſed by Mr. Raphſon to be reſolved, is ſaid to be $- x^5 + 7x^4 - 20x^3 + 155xx = 10{,}000$, or $155xx - 20x^3 + 7x^4 - x^5 = 10{,}000$. But this is not the equation he reſolves; and, indeed, it is not a poſſible equation, becauſe the greateſt poſſible magnitude of the compound quantity $155xx - 20x^3 + 7x^4 - x^5$ is that which it has when the infinitely ſmall increment of the binomial quantity $20x^3 + x^5$ becomes equal to the contemporary increment of the binomial quantity $155xx + 7x^4$, that is, (if we put $\dot{x}$, or x with a point placed over it, for the infinitely ſmall increment of x,) when $20 \times 3x^2\dot{x} + 5x^4\dot{x}$ becomes equal to $155 \times 2x\dot{x} + 7 \times 4x^3\dot{x}$, or when $60x^2 + 5x^4$ is $= 310x + 28x^3$, or when $60x + 5x^3$ is $= 310 + 28xx$, or when

when $5x^3 - 28xx + 60x$ is $= 310$, or when $x^3 - \frac{28xx}{5} + 12x$ is $= 62$, or when $x^3 - 5.6xx + 12x$ is $= 62$; and that is when x is nearly $= 5.5$; at which time the compound quantity $155xx - 20x^3 + 7x^4 - x^5$ will be nearly equal to 2733, as will appear by fubftituting 5.5 inftead of x in the terms of the faid quantity $155xx - 20x^3 + 7x^4 - x^5$: and this quantity 2733 (which is the greateft poffible magnitude of the compound quantity $155xx - 20x^3 + 7x^4 - x^5$,) is very much lefs than 10,000, or the abfolute term of the equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, and confequently the faid equation is impoffible. But Mr. Raphfon, though he fets down this equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, as the equation that is to be refolved, yet really means to refolve a quite different equation, to wit, the equation that refults from fuppofing x to be a negative quantity, or from fubftituting the powers of $-x$, to wit, $+xx$, $-x^3$, $+x^4$, and $-x^5$, in the terms of the faid equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, inftead of the like powers of $+x$, to wit, $+xx$, $+x^3$, $+x^4$, and $+x^5$; by which fubftitution the faid equation will be converted into the equation $155 \times +xx - 20 \times -x^3 + 7 \times +x^4 - 1 \times -x^5 = 10,000$, or $155xx + 20x^3 + 7x^4 + x^5 = 10,000$, which is evidently a poffible equation, and of which the root is 4.544,195,72, or the fame number which he obtains by his folution of the Problem, and which, with the fign $-$ prefixed to it, he calls the negative root of the propofed equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$. Now all this perplexity would have been avoided, if Mr. Raph-

fon

ſon had propoſed at firſt to find the root, or, in the language of modern writers of Algebra, the *affirmative*, or *poſitive*, root, of the equation $155xx + 20x^3 + 7x^4 + x^5 = 10,000$, or $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, which equation is evidently poſſible, and can have only one root. And then all the ſteps of his ſolution would have been clear and eaſy, as will appear by reſolving this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$ according to the principles of his method; which may be done in the manner following:

The Reſolution of the Affected Equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, *by Mr. Raphſon's Method of Approximation.*

Art. 5. In conſidering this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, it is, in the 1ſt place, eaſy to ſee that x muſt be greater than 1. For, if we ſuppoſe x to be $= 1$, we ſhall have $xx = 1$, and $x^3 = 1$, and $x^4 = 1$, and $x^5 = 1$; and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ will be $= 1 + 7 + 20 + 155 = 183$; which is very much leſs than the abſolute term 10,000. Therefore 1 muſt be much leſs than x.

In the ſecond place, if we ſuppoſe x to be $= 10$, we ſhall have $xx = 100$, and $x^3 = 1000$, and $x^4 = 10,000$, and

and $x^5 = 100,000$; so that x^4 alone will be equal to the absolute term 10,000, and consequently $x^5 + 7x^4 + 20x^3 + 155x$ must be very much greater than the said absolute term; and consequently 10 must be much greater than x.

Thirdly, since x is less than 10 and greater than 1, let us suppose it to be equal to 5. Then we shall have $xx = 25$, and $x^3 = 125$, and $x^4 = 625$, and $x^5 = 3125$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 3125 + 7 \times 625 + 20 \times 125 + 155 \times 25 = 3125 + 4375 + 2500 + 3875$) $= 13,875$; which is greater than the absolute term 10,000. Therefore 5 is greater than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

We will therefore, in the 4th place, suppose x to be $= 4$. And then we shall have $xx = 16$, and $x^3 = 64$, and $x^4 = 256$, and $x^5 = 1024$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 1024 + 7 \times 256 + 20 \times 64 + 155 \times 16 = 1024 + 1792 + 1280 + 2480$) $= 6576$; which is less than the absolute term 10,000. Therefore 4 is less than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

It appears therefore that the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$ is greater than 4, but less than 5. And either of these values might very well serve for a first near value of the said root, or for the basis of a further approximation to it. Mr Raphson makes choice of 5, which is greater than the truth.

Art. 6. Let us then suppose a, or the first near value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, to be $= 5$; and let z be the difference by which it exceeds the true value of x. Then will x be $= a - z$, and consequently xx will be $(= \overline{a - z}|^2) = aa - 2az + \&c$, and x^3 will be $(= \overline{a - z}|^3) = a^3 - 3a^2z + \&c$, and x^4 will be $(= \overline{a - z}|^4) = a^4 - 4a^3z + \&c$, and x^5 will be $(= \overline{a - z}|^5) = a^5 - 5a^4z + \&c$. Therefore $x^5 + 7x^4 + 20x^3 + 155xx$ will be $=$

$$\left\{\begin{array}{rrl} & & a^5 - 5a^4z + \&c, \\ + & 7 \times & \overline{a^4 - 4a^3z + \&c,} \\ + & 20 \times & \overline{a^3 - 3a^2z + \&c,} \\ + & 155 \times & \overline{aa - 2az + \&c,} \end{array}\right\}$$

$$= \left\{\begin{array}{rrrl} & a^5 & - & 5a^4z + \&c, \\ + & 7a^4 & - & 28a^3z + \&c, \\ + & 20a^3 & - & 60a^2z + \&c, \\ + & 155aa & - & 310az + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z - 28a^3z - 60a^2z - 310az + \&c$, will also be $= 10{,}000$, and consequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both sides,) we shall have $a^5 + 7a^4 + 20a^3 + 155aa = 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$, or (because a is $= 5$, and consequently $a^5 + 7a^4 + 20a^3 + 155aa$ is $= 13{,}875$, as has been shewn in art. 5,) we shall have $13{,}875 = 10{,}000 + 5a^4z + 28a^3z + 60a^2z$ +

$+ 310az$, and consequently (subtracting 10,000 from both sides,) $3875 = 5a^4z + 28a^3z + 60a^2z + 310az = z \times \overline{5a^4 + 28a^3 + 60a^2 + 310a}$. Therefore z will be $= \frac{3875}{5a^4 + 28a^3 + 60a^2 + 310a}$ ($= \frac{3875}{5 \times \overline{5}|^4 + 28 \times \overline{5}|^3 + 60 \times \overline{5}|^2 + 310 \times 5} = \frac{3875}{5 \times 625 + 28 \times 125 + 60 \times 25 + 310 \times 5} = \frac{3875}{3125 + 3500 + 1500 + 1550} = \frac{3875}{9675}$) $= 0.4$. Therefore $a - z$, or x, will be ($= a - 0.4 = 5.0 - 0.4$) $= 4.6$; and 4.6 will be a second near value of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

We must next try whether this second near value of x is greater or less than it's true value; and for this purpose we must substitute it, instead of x, in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$.

Now, if we suppose x to be $= 4.6$, we shall have xx ($= \overline{4.6}|^2$) $= 21.16$, and x^3 ($= \overline{4.6}|^3$) $= 97.336$, and x^4 ($= \overline{4.6}|^4$) $= 447.7456$, and x^5 ($= \overline{4.6}|^5$) $= 2059.62976$, and $155xx$ ($= 155 \times 21.16$) $= 3279.80$, and $20x^3$ ($= 20 \times 97.336$) $= 1946.720$, and $7x^4$ ($= 7 \times 447.7456$) $= 3134.2192$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 2059.62976 + 3134.2192 + 1946.720 + 3279.80$) $= 10,420.36896$; which is greater than 10,000,

or

or the abſolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Therefore 4.6 will be greater than the true value of x in that equation.

Art. 7. To find a third near value of the root of this equation, let a be ſuppoſed to be $= 4.6$, and z be the difference by which a, or 4.6, exceeds the true value of the ſaid root.

Then we ſhall have, as before, $x = a - z$, and conſequently $xx\ (= \overline{a-z}|^2) = aa - 2az +$ &c, and $x^3\ (= \overline{a-z}|^3) = a^3 - 3a^2z +$ &c, and $x^4\ (= \overline{a-z}|^4) = a^4 - 4a^3z +$ &c, and $x^5\ (= \overline{a-z}|^5) = a^5 - 5a^4z +$ &c, and $7x^4\ (= 7 \times \overline{a^4 - 4a^3z + \&c,}) = 7a^4 - 28a^3z +$ &c, and $20x^3\ (= 20 \times \overline{a^3 - 3a^2z + \&c,}) = 20a^3 - 60a^2z +$ &c, and $155xx\ (= 155 \times \overline{aa - 2az + \&c,}) = 155aa - 310az +$ &c, and $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{\begin{array}{rrrl} & a^5 & - & 5a^4z + \&c, \\ + & 7a^4 & - & 28a^3z + \&c, \\ + & 20a^3 & - & 60a^2z + \&c, \\ + & 155aa & - & 310az + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10,000$,

Therefore

$$\left\{\begin{array}{rrrl} & a^5 & - & 5a^4z + \&c, \\ + & 7a^4 & - & 28a^3z + \&c, \\ + & 20a^3 & - & 60a^2z + \&c, \\ + & 155aa & - & 310az + \&c, \end{array}\right\} \text{ will likewiſe be}$$

=

$= 10{,}000$, and consequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both sides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been shewn in the last article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $\overline{4.6}|^5 + 7 \times \overline{4.6}|^4 + 20 \times \overline{4.6}|^3 + 155 \times \overline{4.6}|^2$, is $= 10{,}420.36896$.

Therefore $10{,}420.36896$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and consequently (subtracting $10{,}000$ from both sides of the equation,) 420.36896 will be $= 5a^4z + 28a^3z + 60a^2z + 310az$ ($= 5 \times \overline{4.6}|^4 \times z + 28 \times \overline{4.6}|^3 \times z + 60 \times \overline{4.6}|^2 \times z + 310 \times 4.6 \times z = 5 \times 447.7456 \times z + 28 \times 97.336 \times z + 60 \times 21.16 \times z + 310 \times 4.6 \times z = 2238.7280 \times z + 2725.408 \times z + 1269.60 \times z + 1426.0 \times z$) $= 7659.7360 \times z$, and consequently z will be ($= \frac{420.36896}{7659.7360}$) $= 0.0548$, or nearly 0.055. Therefore x, or $a - z$, or $4.6 - z$, will be nearly ($= 4.6 - 0.055$,) $= 4.545$; and consequently this number 4.545 will be a third near value of the root of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Q. E. I.

Now let this number 4.545 be substituted instead of x in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$, in order to discover whether the result will be greater, or less,

less, than 10,000, or the absolute term of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Now, if x be supposed to be $= 4.545$, we shall have xx $(= \overline{4.545}|^2) = 20.657,025$, and x^3 $(= \overline{4.545}|^3) = 93.886,178,625$, and x^4 $(= \overline{4.545}|^4) = 426.712,681,850,625$, and x^5 $(= \overline{4.545}|^5) = 1939.409,139,011,090,625$, and consequently $7x^4$ $(= 7 \times 426.712,681,850,625) = 2986.988,772,954,375$, and $20x^3$ $(= 20 \times 93.886,178,625) = 1877.723,572,500$, and $155xx$ $(= 155 \times 20.657,025) = 3201.838,875$, and $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 1939.409,139,011,090,625 + 2986.988,772,954,375 + 1877.723,572,500 + 3201.838,875) = 10,005.960,359,465,465,625$; which is greater than 10,000, or the absolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Therefore 4.545 will be greater than the true value of x in that equation.

Art. 8. To find a fourth near value of the root of this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, let a be supposed to be $= 4.545$, and z be supposed to be the difference by which a, or 4.545, exceeds the true value of the said root.

Then we shall, as before, have $x = a - z$, and consequently xx $(= \overline{a - z}|^2) = aa - 2az + \&c$, and x^3 $(= \overline{a - z}|^3) = a^3 - 3a^2z + \&c$, and x^4 $(= \overline{a - z}|^4) = a^4 - 4a^3z + \&c$, and x^5 $(= \overline{a - z}|^5) = a^5 - 5a^4z + \&c$, and

and $7x^4$ $(= 7 \times \overline{a^4 - 4a^3z + \&c,}) = 7a^4 - 28a^3z + \&c,$ and $20x^3$ $(= 20 \times \overline{a^3 - 3a^2z + \&c,}) = 20a^3 - 60a^2z + \&c,$ and $155xx$ $(= 155 \times \overline{aa - 2az + \&c,}) = 155aa - 310az + \&c,$ and consequently $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{\begin{array}{rrrrl} & a^5 & - & 5a^4z & + \&c, \\ + & 7a^4 & - & 28a^3z & + \&c, \\ + & 20a^3 & - & 60a^2z & + \&c, \\ + & 155a^2 & - & 310az & + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z + \&c, - 28a^3z + \&c, - 60a^2z + \&c, - 310az + \&c,$ will likewise be $= 10{,}000$; and consequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both sides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been shewn in the last article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $\overline{4.545}|^5 + 7 \times \overline{4.545}|^4 + 20 \times \overline{4.545}|^3 + 155 \times \overline{4.545}|^2$, is $= 10{,}005.960{,}359{,}465{,}465{,}625$.

Therefore $10{,}005.960{,}359{,}465{,}465{,}625$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and consequently (subtracting 10,000 from both sides,) $5.960{,}359{,}465{,}465{,}625$ will be $= 5a^4z + 28a^3z + 60a^2z + 310az$

$310az$ ($= 5 \times \overline{4.545}|^{4} \times z + 28 \times \overline{4.545}|^{3} \times z + 60 \times \overline{4.545}|^{2} \times z + 310 \times 4.545 \times z = 5 \times 426.712,681,850,625 \times z + 28 \times 93.886,178,625 \times z + 60 \times 20.657,025 \times z + 310 \times 4.545 \times z = 2133.563,409,253,125 \times z + 2628.813,001,500 \times z + 1239.421,500 \times z + 1408.950 \times z$) $= 7410.747,910,753,125 \times z$. Therefore z will be ($= \frac{5.960,350,465,465,625}{7410.747,910,753,125}$) $= 0.000,804,28$, and x, or $a - z$, or $4.545 - z$, will be ($= 4.545,000,00 - 0.000,804,28$) $= 4.544,195,72$. Therefore 4.544,195,72 will be a fourth near value of the root of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

This number 4.544,195,72, agrees with the number found by Mr. Raphson, in all it's figures.

Art. 9. The foregoing resolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, has been performed at great length, in order to set forth, in as clear a manner as possible, the several reasonings upon which the arithmetical operations used in it are grounded, as well as the said operations themselves. And by so doing the subject is rendered so much easier than in Mr. Raphson's very concise and compressed way of treating it, (in which all the reasonings are dropped, and only the arithmetical operations are exhibited,) that, though the above resolution of the said equation is three, or four, times as long as Mr. Raphson's, yet I am fully persuaded that it may be read and understood in a third, or fourth, part of the time

time that is neceffary to a thorough comprehenfion of Mr. Raphfon's refolution of it; even if he had not puzzled the matter by talking of the negative root of the equation $-x^5 + 7x^4 - 20x^3 + 155xx = 10{,}000$. But that this may appear the more clearly, I will now repeat the foregoing refolution of this equation in the ftyle and manner of Mr. Raphfon, by omitting the feveral reafonings fet forth in the foregoing articles, and making ufe of a Canon, or Theorem, for the purpofe of computing the fecond, third, and fourth values of z, in the fame manner as Mr. Raphfon has done.

Art. 10. Since each of the three firft fucceffive near values of x, or the root of the propofed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, from which the next near values of it are derived, to wit, the three numbers 5, 4.6, and 4.545, and which are fucceffively denoted by the letter a, is greater than the true value of x in the faid equation, or than the root of the faid equation, it follows that the fecond, and third, and fourth near values of x will, each of them, be fucceffively denoted by the refidual quantity $a - z$; and confequently, by applying the reafonings ufed in art. 6, in order to obtain the values of z, and of $a - z$, or x, we fhall find that z will be, fucceffively, nearly equal to the value of the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$, and, therefore, that $a - z$, or x, will be, fucceffively, nearly equal to the value of the quantity a — the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155a^2 - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$. This, then, is the Theorem,

Theorem, or Canon, by the application of which we are to compute the firſt, and ſecond, and third near values of $a - z$, or the ſecond, and third, and fourth near values of x, after taking 5 for the firſt near value of it, or for the firſt value of a.

Now, if a is $= 5$, we ſhall have $z =$ the fraction

$$\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 20a^3 + 60a^2 + 310a} = \frac{3875}{9675} = 0.4.$$

Therefore $a - z$ will be $(= 5 - 0.4) = 4.6$; which will therefore be the ſecond near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Secondly, if a be $= 4.6$, we ſhall have $z =$

$$\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{420.36896}{7659.7360} =$$

0.0548, or, nearly, 0.055. Therefore $a - z$ will be $(= 4.6 - 0.055) = 4.545$; which will therefore be the third near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Thirdly, if a be $= 4.545$, we ſhall have $z =$

$$\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{5.960{,}359.465{,}465{,}625}{7410.747{,}910{,}753{,}125}$$

$= 0.000{,}804{,}28$. Therefore $a - z$ will be $(= 4.545 - 0.000{,}804{,}28) = 4.544{,}195{,}72$; which will therefore be the fourth near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Q. E. I.

Art. 11.

Art. 11. Mr. Raphſon's Canon, or Theorem, for the value of z, is expreſſed more conciſely than the foregoing Theorem, $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$. For he uſes the letters b, c, d, and f, for the co-efficients 7, 20, and 155, of the fourth, third, and ſecond, power of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, and for 10,000, the abſolute term of that equation, reſpectively; which produces the following Canon, or Theorem, for the value of z, to wit, $z = \frac{a^5 + ba^4 + ca^3 + da^2 - f}{5a^4 + 4ba^3 + 3ca^2 + 2da}$. But it appears to me that, though we may ſeem to gain ſomething in point of brevity by uſing this very general notation, we loſe as much in the article of perſpicuity, which is a matter of much greater importance. However, this latter reſolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, which is expreſſed in Mr. Raphſon's conciſe ſtyle and manner, and the foregoing more explicit reſolution of it in art. 5, 6, 7, and 8, (in which the reaſonings, on which the ſeveral arithmetical operations are grounded, are diſtinctly ſet forth and repeated,) are, both of them, the ſame in ſubſtance, and are, as I believe, the very beſt method that can be taken for diſcovering the root of the ſaid equation.

Of the Reſemblance of Mr. Raphſon's Method of Reſolving Numeral Equations by Approximation to Sir Iſaac Newton's Method of doing the ſame thing.

Art. 12. It has been obſerved above in art. 2, that Sir Iſaac Newton's method of reſolving numeral equations by

by approximation differed but little from Mr. Raphson's, both methods being founded on the same principle of considering the new, or transformed, equation, (resulting from the substitution of $a + z$, or $a - z$, instead of x, in the original equation,) as a mere simple equation, or neglecting, or omitting, all the terms of it which involved in them any higher power of z than it's simple power; which reduces the resolution of all equations, of whatever orders, to the resolution of a simple equation, or, rather, to the resolution of several successive simple equations, by which we make continual approaches to the true value of the root of the original equation. In this grand principle Sir Isaac Newton's method and Mr. Raphson's method perfectly agree; and, in finding the second near value of x, or in making the first approximation to the true value of x, after having obtained, by conjecture, or trial, or in some other manner, the value of what has been here called a, or a first near value of x, or the root sought, there is not the smallest difference between them. But in the investigation of the third, and fourth, and other following near values of x, there is a little difference in their manner of proceeding, which the reader may be glad to see examined. I shall therefore now compare the two methods together, in the case of a very easy equation, by which Sir Isaac Newton himself has thought proper to illustrate his method.

A COM-

A COMPARISON BETWEEN SIR ISAAC NEWTON'S AND MR. RAPHSON'S METHODS OF RESOLVING NUMERAL EQUATIONS BY APPROXIMATION.

Art. 13. SIR ISAAC NEWTON'S method of refolving numeral equations by approximation, is explained by himfelf in his curious little Tract, intitled, *Analyfis per Æquationes Numero Terminorum Infinitas*, (which was written in the year 1666, and communicated to Dr. Ifaac Barrow, and to Mr. John Collins, and to other learned men of that time, in the year 1669,) by an example; which is as follows:

Art. 14. Let it be required to refolve the cubick equation $x^3 - 2x = 5$.

Here, in the firft place, it is eafy to fee that x is fomewhat greater than 2, but much lefs than 3. For, if x is taken equal to 2, we fhall have $2x = 4$, and $x^3 = 8$, and confequently $x^3 - 2x$ $(= 8 - 4) = 4$; which is lefs than 5, or the true value of $x^3 - 2x$ in the propofed equation: and, if x is taken equal to 3, we fhall have $2x = 6$, and $x^3 = 27$, and confequently $x^3 - 2x$ $(= 27 - 6) = 21$; which is very much greater than 5, or the true value of $x^3 - 2x$ in the propofed equation. Therefore the true value of x in that equation muft be

X much

much leſs than 3, and a little greater than 2. Let it therefore be ſuppoſed to be equal to the quantity $2 + z$, in which z denotes the unknown quantity by which the true value of x exceeds 2. And let $2 + z$ be ſubſtituted, inſtead of x, in the propoſed equation $x^3 - 2x = 5$. This may be done as follows:

Since x is $= 2 + z$, we ſhall have x^3 $(= \overline{2}|^3 + 3 \times \overline{2}|^2 \times z + 3 \times 2 \times zz + z^3 = 8 + 3 \times 4 \times z + 3 \times 2zz + z^3) = 8 + 12z + 6zz + z^3$, and $2x$ $(= 2 \times \overline{2 + z}) = 4 + 2z$, and conſequently $x^3 - 2x$ $(= 8 + 12z + 6zz + z^3 - 4 - 2z) = 4 + 10z + 6zz + z^3$. But $x^3 - 2x$ is $= 5$. Therefore $4 + 10z + 6zz + z^3$ will alſo be $= 5$, and conſequently (ſubtracting 4 from both ſides,) $10z + 6zz + z^3$ will be $= 1$; and, (ſubtracting $6zz + z^3$ from both ſides,) $10z$ will be $= 1 - 6zz - z^3$. Therefore z will be $= \frac{1 - 6zz - z^3}{10}$ $= \frac{1}{10} - \overline{\frac{6zz + z^3}{10}} = 0.1 \frac{- 6zz - z^3}{10}$, that is, z is leſs than $\frac{1}{10}$, or 0.1, by the quantity $\frac{6zz + z^3}{10}$. Therefore x, or $2 + z$, is leſs than $2 + \frac{1}{10}$, or $2 + 0.1$, or 2.1, by the ſaid quantity $\frac{6zz + z^3}{10}$; which, on account of the ſmallneſs of z, (which is leſs than $\frac{1}{10}$,) will be a very ſmall quantity in compariſon of z, or of

of $\frac{1}{10}$, and, *à fortiori*, in comparifon of 2, and confequently may be neglected. And therefore 2.1 will be a fecond near value of x, or the root of the propofed equation $x^3 - 2x = 5$, that will be a little greater than it's true value, but nearer to it than any other number that confifts of only two places of figures. Q. E. I.

This is the firft ftep of Sir Ifaac Newton's approximation to the root of the equation $x^3 - 2x = 5$, after the affumption of the number 2, by conjecture and trial, for it's firft near value. And in this firft ftep of the approximation Sir Ifaac Newton's and Mr. Raphfon's methods exactly co-incide.

Art. 15. But in the next ftep of the approximation to the value x, in the faid equation $x^3 - 2x = 5$, the two methods are fomewhat different from each other, though the number of new figures of the true value of x, that are exact in the next near values of it refulting from both methods, is the fame. The difference between the methods in this fecond ftage of the approximation is as follows:

Mr. Raphfon corrects the value of x, or the root of the original equation $x^3 - 2x = 5$, already found, to wit, 2.1, (and which is known to be fomewhat greater than the truth,) by fubtracting from it the unknown quantity by which it exceeds x, and which we may call v, and fubftituting $2.1 - v$ inftead of x in the faid original equation, $x^3 - 2x = 5$, whereby it is transformed into

another cubick equation, in which v will be the only unknown quantity; and then he finds a near value of v by resolving the said transformed equation as if it were only a simple equation, or by neglecting the terms which involve the square and cube of v, on account of their smallness, just as we before neglected the terms $6zz$ and z^3 in the foregoing transformed equation $10z + 6zz + z^3 = 1$ for the same reason. But Sir Isaac Newton takes no further notice of the original equation $x^3 - 2x = 5$, till he has compleated the whole process of his approximation; but, instead of the said original equation, he considers the former transformed equation, $10z + 6zz + z^3 = 1$, which was derived from it, and investigates the value of it's root, z, to a greater degree of exactness than that to which it was before obtained. And this he does in the manner following:

Since it has been seen that z is less than 0.1, let the quantity by which 0.1 exceeds it be called v, so that z shall be $= 0.1 - v$; and let $0.1 - v$ be substituted, instead of z in the transformed equation $10z + 6zz + z^3 = 1$. This may be done as follows:

Since z is $= 0.1 - v$, we shall have

$$zz \;(= \overline{0.1 - v}|^2) = 0.01 - 0.2v + vv,$$

and $$z^3 \;(= \overline{0.1 - v}|^3 = 0.001 - 3 \times 0.01 \times v + 3 \times 0.1 \times vv - v^3) = 0.001 - 0.03v + 0.3vv - v^3,$$

and $$10z \;(= 10 \times \overline{0.1 - v}) = 1 - 10v,$$

and $$6zz \;(= 6 \times \overline{0.01 - 0.2v + vv}) = 0.06 - 1.2v + 6vv,$$

and consequently

$$\left\{\begin{array}{l} 10z \\ +\ 6zz \\ +\ z^3 \end{array}\right\} = \left\{\begin{array}{llll} 1.00 & -\ 10.00v & & \\ +\ 0.06 & -\ 1.20v & +\ 6vv & \\ +\ 0.001 & -\ 0.03v & +\ 0.3vv & -\ v^3 \end{array}\right\}$$

$$= 1.061 - 11.23v + 6.3vv - v^3.$$

But $10z + 6zz + z^3$ is $= 1$.

Therefore $1.061 - 11.23v + 6.3vv - v^3$ will likewife be $= 1$. And confequently (adding $11.23v$ to both fides,) we fhall have $1.061 + 6.3vv - v^3 = 1 + 11.23v$; and, (fubtracting 1 from both fides,) we fhall have $0.061 + 6.3vv - v^3 = 11.23v$, and (neglecting $6.3vv$ and v^3 as inconfiderable in comparifon of 0.061 and $11.23v$) we fhall have $0.061 = 11.23v$, or $11.23v = 0.061$; and confequently (dividing both fides by 11.23,) we fhall have $v \left(= \frac{0.061}{11.23}\right) = 0.0054$. Therefore z, or $0.1 - v$, will be $(= 0.1 - 0.0054) = 0.0946$, and confequently x, or $2 + z$, will be $(= 2 + 0.0946) = 2.0946$.

Q. E. I.

In this manner Sir Ifaac Newton finds the root of the propofed equation $x^3 - 2x = 5$ to be equal to 2.0946, which is as near the truth as five figures can exprefs it.

Art. 16. He then carries the inveftigation one ftep further, by which he obtains the value of x exact to nine places of figures; and for this purpofe he proceeds in the manner following:

The laſt transformed equation was $11.23v = 0.061 + 6.3vv - v^3$; from which it follows that v is accurately equal to $\frac{0.061}{11.23} + \frac{6.3vv - v^3}{11.23}$, or $0.0054 + \frac{6.3vv - v^3}{11.23}$, which is greater than 0.0054 alone, becauſe $6.3vv$ is greater than v^3. Since, therefore, v is greater than 0.0054, let us ſuppoſe it to be $= 0.0054 + w$; and let this binomial quantity be ſubſtituted, inſtead of v, in the transformed equation $11.23v = 0.061 + 6.3vv - v^3$, or, rather, in the equation $11.23v - 6.3vv + v^3 = 0.061$, conſiſting of the ſame terms as the former, but in which the terms involving the unknown quantity v are all brought to the ſame ſide of the equation, and ranged according to the powers of v, beginning from it's loweſt power, or the ſimple power of v. This may be done in the manner following:

Since v is $= 0.0054 + w$, we ſhall have

$$vv\ (= \overline{0.0054+w}|^2 = \overline{0.0054}|^2 + 2 \times 0.0054 \times w + w^2)$$
$$= 0.000,029,16 + 0.0108 \times w + w^2,$$

$$\text{and } v^3\ (= \overline{0.0054+w}|^3 = \overline{0.0054}|^3 + 3 \times \overline{0.0054}|^2 \times w + 3 \times 0.0054 \times w^2 + w^3$$
$$= 0.000,000,157,464 + 3 \times 0.000,029,16 \times w + 0.0162 \times w^2 + w^3)$$
$$= 0.000,000,157,464 + 0.000,087,48 \times w + 0.0162 \times w^2 + w^3,$$

$$\text{and } 11.23v\ (= 11.23 \times \overline{0.0054 + w}) = 0.060,642 + 11.23 \times w,$$

$$\text{and } 6.3vv\ (= 6.3 \times \overline{0.000,029,16 + 0.0108 \times w + w^2})$$
$$= 0.000,183,708 + 0.068,04 \times w + 6.3ww;$$

and

and consequently $11.23v - 6.3vv + v^3$ will be =

$$\left\{\begin{array}{l} 0.060,642 + 11.23 \times w \\ -0.000,183,708 - 0.068,04 \times w - 6.3 \times ww. \\ +0.000,000,157,464 + 0.000,087,48 \times w + 0.0162w^2 + w^3) \end{array}\right\}$$

$$= \left\{\begin{array}{llll} 0.060,642,157,464 & +11.230,087,48 \times w & +0.0162w^2 & +w^3 \\ -0.000,183,708 & -0.068,04 \times w & -6.3 \times w^2 & \end{array}\right\}$$

$$= 0.060,458,449,464 + 11.162,047,48w - 6.2838w^2 + w^3.$$

But $11.23v - 6.3vv + v^3$ is = 0.061.

Therefore $0.060,458,449,464 + 11.162,047,48 \times w - 6.2838 \times ww + w^3$ will likewise be = 0.061; and consequently (subtracting 0.060,458,449,464 from both sides,) $11.162,047,48 \times w - 6.2838ww + w^3$ will be (= 0.061,000,000,000 — 0.060,458,449,464) = 0.000, 541,550,536; and (neglecting the terms $6.2838ww$ and w^3, as inconsiderable in comparison of $11.162,047,48 \times w$,) we shall have $11.162,047,48 \times w = 0.000,541,550,536$, and consequently w $(= \frac{0.000,541,550,536}{11.162,047,48}) =$ 0.000,048,52. Therefore v, or $0.0054 + w$, will be (= 0.0054 + 0.000,048,52) = 0.005,448,52, and z, or $0.1 - v$, will be (= 0.100,000,00 — 0.005,448,58) = 0.094,551,48, and x, or $2 + z$, will be (= 2 + 0.094, 551,48) = 2.094,551,48; that is, the root of the proposed equation $x^3 - 2x = 5$ will be = 2.094,551,48.

Q. E. I.

This number 2.094,551,48 is exact in all the figures, as will be shewn in a subsequent Article.

Art. 17. Having thus ſet forth Sir Iſaac Newton's method of inveſtigating the root of the propoſed equation $x^3 - 2x = 5$ to nine places of figures, we muſt now perform the ſame thing by Mr. Raphſon's method, in order to make a compariſon between the neceſſary operations of the two methods.

Now Mr. Raphſon's method of approximating further to the root of the equation $x^3 - 2x = 5$, after having found it to be equal to $2 + 0.1 - \frac{6zz - z^3}{10}$, or to be ſomewhat leſs than 2.1, is to put v for the unknown quantity by which it falls ſhort of 2.1, and then to ſubſtitute the reſidual quantity $2.1 - v$ in the terms of the original equation $x^3 - 2x = 5$, whereby the ſaid equation will be transformed into another cubick equation, in which v will be the only unknown quantity; and then he determines the value of v by reſolving the ſaid transformed equation as if it was a mere ſimple equation, or by neglecting the terms in which the ſquare or the cube of v occur. This may be done in the manner following:

Since x is $= 2.1 - v$, we ſhall have

$$xx \,(= \overline{2.1 - v}|^2 = \overline{2.1}|^2 - 2 \times 2.1 \times v + \&c)$$
$$= 4.41 - 4.2v + \&c,$$
$$\text{and } x^3 \,(= \overline{2.1 - v}|^3 = \overline{2.1}|^3 - 3 \times \overline{2.1}|^2 \times v + \&c$$
$$= 9.261 - 3 \times 4.41 \times v + \&c)$$
$$= 9.261 - 13.23 \times v + \&c,$$

and

and $2x$ ($= 2 \times \overline{2.1 - v}$) $= 4.2 - 2v$,

and consequently $x^3 - 2x$ ($= 9.261 - 13.23 \times v +$ &c $- 4.2 + 2v$) $= 5.061 - 11.23 \times v$ &c.

But $x^3 - 2x$ is $= 5$.

Therefore $5.061 - 11.23 \times v$ &c, will likewise be $= 5$, and consequently (adding $11.23 \times v$ to both sides,) we shall have $5.061 = 5 + 11.23 \times v$, and (subtracting 5 from both sides,) we shall have $11.23 \times v = 0.061$, and consequently v ($= \frac{0.061}{11.23}$) $= 0.0054$. Therefore x, or $2.1 - v$, will be ($= 2.1 - 0.0054$) $= 2.0946$; or 2,0946 will be a third near value of the root of the proposed equation $x^3 - 2x = 5$. Q. E. I.

This third near value of x is the very same with the third near value of it obtained above, in art. 15, by Sir Isaac Newton's method.

Art. 18. In this step of the approximation, by which we obtain the number 2.0946 for the third near value of the root of the proposed equation $x^3 - 2x = 5$, the principal difference between the two methods seems to consist in this, to wit, that by Mr. Raphson's method we are obliged to raise the two first terms of the powers of the compound quantity $2.1 - v$, and consequently to raise the powers of the number 2 1, which consists of two figures; whereas in Sir Isaac Newton's method of proceeding, we had occasion only to raise the powers of

the compound quantity $0.1 - v$, and consequently to raise the powers of the number 0.1, which consists of only one figure; which is somewhat easier than to raise the powers of 2.1. But both operations are so easy, that the difference of the labour of performing them is hardly worth considering. And, with respect to the simplicity of conception in the two methods, Mr. Raphson's method seems to be preferable to Sir Isaac Newton's, because the former always refers to the original equation $x^3 - 2x = 5$, whereas the latter method refers to the preceeding transformed equation $10z + 6zz + z^3 = 1$, which has more terms and larger co-efficients than the original equation $x^3 - 2x = 5$.

Art. 19. But in the next step of the approximation by Mr. Raphson's method, we shall find the labour of raising the powers of the value of x already found, to wit, the powers of 2.0946, to be considerably greater than that of raising the powers of the last preceeding supplement of it according to Sir Isaac Newton's method, that supplement being only the decimal fraction 0.0054, in which there are only two significant figures. This will appear by performing this step of the approximation by Mr. Raphson's method; which may be done as follows:

Art. 20. The last near value we found for x, or the root of the equation $x^3 - 2x = 5$, by Mr. Raphson's method, was 2.0946. Now this near value of x is greater than it's true value. For, if we suppose x to be $= 2.0946$, we shall have x^3 $(= \overline{2.0946}|^3) = 9.189,741,550,536$, and

2x

$2x$ $(= 2 \times 2.0946) = 4.1892$, and consequently $x^3 - 2x$ $(= 9.189,741,550,536 - 4.1892) + 5.000,541,550,536$; which is greater than 5, or the absolute term of the equation $x^3 - 2x = 5$: and consequently 2.0946 must be greater than the true value of the root of the said equation.

We will therefore suppose x to be $= 2.0946 - w$, and substitute this residual quantity instead of x in the terms of the equation $x^3 - 2x = 5$.

Now, since x is $= 2.0946 - w$, we shall have

$$xx\ (= \overline{2.0946 - w}|^2 = \overline{2.0946}|^2 - 2 \times 2.0946 \times w + \&c)$$
$$= 4.387,349,16 - 4.1892 \times w + \&c,$$
$$\text{and } x^3\ (= \overline{2.0946 - w}|^3 = \overline{2.0946}|^3 - 3 \times \overline{2.0946}|^2 \times w + \&c$$
$$= 9.189,741,550,536 - 3 \times 4.387,349,16 \times w + \&c)$$
$$= 9.189,741,550,536 - 13.162,047,48 \times w + \&c,$$
$$\text{and } 2x\ (= 2 \times \overline{2.0946 - w}) = 4.1892 - 2w,$$

and consequently $x^3 - 2x =$

$$\left\{ \begin{array}{llll} & 9.189,741,550,536 & - & 13.162,047,48 \times w + \&c \\ - & 4.189,2 & + & 2.000,000,00 \times w \end{array} \right\}$$
$$= 5.000,541,550,536 - 11.162,047,48 \times w + \&c.$$

But $x^3 - 2x$ is $= 5$.

Therefore $5.000,541,550,536 - 11.162,047,48 \times w$ $+ \&c$, will be $= 5$; and consequently (adding 11.162, 047,48

047,48 $\times w$ to both ſides,) we ſhall have 5.000,541,550,536 = 5 + 11.162,047,48 $\times w$, and (ſubtracting 5 from both ſides,) 0.000,541,550,536 = 11.162,047,48 $\times w$, or 11.162,047,48 $\times w$ = 0.000,541,550,536. Therefore w will be $\left(= \frac{0.000,541,550,536}{11.162,047.48}\right)$ 0.000,048,52; and conſequently x, or 2.0946 — w, will be (= 2.094,600,00 — 0.000,048,52) = 2.094,551,48. Therefore 2.094,551,48 will be a fourth near value of x, or the root of the propoſed equation $x^3 - 2x = 5$.

Q. E. I.

This fourth near value of x is the very ſame with the fourth near value of it obtained above, in art. 16, by Sir Iſaac Newton's method.

Art. 21. In this laſt ſtage of Mr. Raphſon's approximation to the root of the propoſed equation $x^3 - 2x = 5$, we have been obliged to raiſe the powers of the number 2.0946, which conſiſts of five places of figures; whereas in Sir Iſaac Newton's way of proceeding we only raiſed the powers of the decimal fraction 0.0054, which contains only two ſignificant figures. But then in that way of proceeding we were obliged to multiply v, or 0.0054 + w, into 11.23, and vv, or the trinomial quantity 0.000,029,16 + 0.0108 $\times w + w^2$, into 6.3; whereas in Mr. Raphſon's way of proceeding we have only to multiply x, or the binomial quantity 2.0946 — w, into the very ſimple co-efficient 2. So that, upon the whole, the difference of the labour of computation in the two methods is not very conſiderable, though it is rather

rather lefs in Sir Ifaac Newton's method than in Mr. Raphfon's. But in point of fimplicity of conception Mr. Raphfon's method feems much fuperiour to Sir Ifaac's, becaufe it never lofes fight of the original equation $x^3 - 2x = 5$, which is to be refolved.

And, further, we may obferve, in favour of Mr. Raphfon's method, that it never requires us to raife any more than the two firft terms of the binomial and refidual quantities $2 + z$, and $2.1 - v$, and $2.0946 - w$, which are fubftituted inftead of x in the original equation $x^3 - 2x = 5$; whereas in Sir Ifaac Newton's method it is neceffary to raife the other terms of the binomial and refidual quantities $2 + z$, and $0.1 - v$, and $0.0054 + w$; which increafes the number and intricacy of the operations of the inveftigation. And therefore, upon the whole, I confider Mr. Raphfon's method of approximating to the values of the roots of fuch equations as preferable to Sir Ifaac Newton's.

A Proof of the Exactnefs of the Number 2.094,551,48, *that has been found by the foregoing Methods of Approximation for the Root of the Equation* $x^3 - 2x = 5$.

Art. 22. It remains that we prove the work to have been rightly performed, or that we fhew that the laft number

number 2.094,551,48, obtained by both these methods, is a very near value of the root x of the proposed equation $x^3 - 2x = 5$, and that we determine to how many figures it is exact.

Now the plainest and best method of doing this is to substitute the number 2.094,551,48, instead of x, in the compound quantity $x^3 - 2x$, in order to discover whether the quantity resulting from this substitution will be greater, or less, than 5, or the absolute term of the proposed equation $x^3 - 2x = 5$: and, if it shall appear that the said result is greater than 5, we may conclude that the said number 2.094,551,48 is greater than the true value of x in the said equation; and, if it shall appear that the said result is less than 5, we may conclude that the said number is less than the true value of x. And, when this has been thus discovered, we must, in the next place, endeavour to determine to how many figures this number 2.094,551,48 co-incides with the more accurate value of x: and, for this purpose, we must, if this number be less than x, increase it by the addition of an unit in the last place of figures; and, if it be greater than x, we must diminish it by the same small quantity, and then substitute the new number thereby obtained, to wit, 2.094,551,49, or 2.094,551,47, instead of x, in the compound quantity $x^3 - 2x$. And, if it shall appear that the value of that compound quantity resulting from such substitution is greater, or less, than 5, we may conclude that the number 2.094,551,49, or 2.094,551,47, is accordingly greater,

greater, or lefs, than the true value of x, in the equation $x^3 - 2x = 5$, and confequently that the faid true value is of an intermediate magnitude between 2.094, 551,49 and 2.094,551,48, or between 2.094,551,48 and 2.094,551,47.

Now, if we take $x = 2.094,551,48$, we fhall have

$$xx = 4.387,145,902,370,190,4,$$
$$\text{and } x^3 = 9.189,102,942,785,417,810,201,792,$$
$$\text{and } 2x = 4.189,102,96,$$

and confequently

$$x^3 - 2x = 4.999,999,982,785,417,810,201,792;$$

which number is fomewhat lefs than 5, or the abfolute term of the propofed equation $x^3 - 2x = 5$. Therefore 2.094,551,48 muft be fomewhat lefs than the true value of x in the faid equation.

Secondly, fince x is greater than 2.094,551,48, we muft now compare it with 2.094,551,49, by fubftituting that number inftead of it in the compound quantity $x^3 - 2x$.

Now, if x is taken $= 2.094,551,49$, or $2.094,551,48 + 0.000,000,01$, we fhall have x^3 $(= \overline{2.094,551,48}|^3 + 3 \times \overline{2.094,551,48}|^2 \times \overline{0.000,000,01}| + 3 \times 2.094,551,48 \times \overline{0.000,000,01}|^2 + \overline{0.000,000,01}|^3$

=

$= 9.189,102,942$, &c, $+ 3 \times 4.387,145,902$, &c $\times 0.000,000,01 + 0.000,000,000$, &c $+ 0.000,000,000$, &c $= 9.189,102,942$, &c $+ 13.161,437,706$, &c $\times 0.000,000,01 + 0.000,000,000$ &c $+ 0.000,000,000$, &c $= 9.189,102,942$, &c $+ 0.000,000,131$, &c $+ 0.000,000,000$, &c $+ 0.000,000,000$, &c) $= 9.189,103,073$, &c; and $2x$ ($= 2 \times 2.094,551,49$) $= 4.189,102,98$; and confequently $x^3 - 2x$ ($= 9.189,103,07$, &c $- 4.189,102,98$) $= 5.000,000,09$, &c; which is greater than 5. Therefore 2.094,551,49 muft be greater than the true value of x in the equation $x^3 - 2x = 5$.

But it has been fhewn that 2.094,551,48 is lefs than the faid true value.

Therefore the true value of x in the equation $x^3 - 2x = 5$, will be of an intermediate magnitude between 2.094,551,48 and 2.094,551,49; and confequently all the figures of the number 2.094,551,48, which we found by the foregoing proceffes of Sir Ifaac Newton's and Mr. Raphfon's methods of approximation for a fourth near value of the root of the equation $x^3 - 2x = 5$, are exact. Q. E. D.

Of

Of the Difficulty of finding a, *or the First Near Value of the Root of an Affected Equation, in certain Cases.*

Art. 23. There is another difficulty that occurs sometimes in resolving high equations by approximation, whether by Sir Isaac Newton's method or by Mr. Raphson's; which indeed are substantially the same. The difficulty I mean, is that of finding the first near value of the root sought (which we have called a in this discourse,) to one, or two places of figures, in order to make it the basis of a further approximation to the true value of the root by either of these methods of approximation. Now, when the equation is known to have but one root, that is, but one real and affirmative root, (for all other roots are not worth considering,) this difficulty will not be great; because it will always be easy to find a tolerably near value of the root by conjectures and trials, and particularly by supposing x, or the root of the proposed equation, first to be equal to 1, and 2dly, to be = 10, and 3dly, to be equal to some short intermediate number between 1 and 10, consisting of only one figure, or, if the root appears to be greater than 10, by supposing it to be equal to 100, or 1000, and afterwards supposing it to be equal to some short intermediate number consisting of two figures; as was done above in art. 5, in finding the first near value of x in the equation $x^5 + 7x^4 + 20x^3 +$

$155xx = 10,000$. But, when the equation confifts of terms connected together partly by the fign $+$, and partly by the fign $-$, and confequently it may, for aught we know to the contrary, have two, or three, or four, or more real and affirmative roots, which may be of very different magnitudes, the aforefaid method of conjectures and trials (though by no means ufelefs,) is lefs expeditious and fatisfactory in affifting us to find the firft near value of one of the roots than in the former cafe; and we are often puzzled to know which of the roots it would be moft expedient to begin to inveftigate. Now, in moft of thefe cafes, I believe, it will be advifeable to begin by inveftigating the leaft root, and for that purpofe to expunge from the equation all the terms that have the fign $-$ prefixed to them, and to find, to about two places, or, at moft, to three places, of figures, the root of the remaining equation. For this root will always be lefs than the leaft root of the original equation, if it really has (as it appears to have,) more than one real and affirmative root; or it will be lefs than the only root of the original equation, if (notwithftanding the appearances to the contrary,) it really has but one root. When the root of this fecond, or curtailed, equation, has been difcovered, it may be called a, and made the ground-work of an approximation to the leaft root of the original equation, and the binomial quantity $a + z$ may be fubftituted in the original equation inftead of x, and the transformed equation thence arifing may be refolved as if it was a mere fimple equation, agreeably to Mr. Raphfon's method of approximation; and the value of z thereby obtained, being added to a, will give us a known value

value of $a + z$, or a ſecond near value of the leaſt, or the only, root of the propoſed equation: after which we may proceed to find the ſaid leaſt, or only, root of the propoſed equation by a further proſecution of Mr. Raphſon's method of approximation above-deſcribed, untill we have determined it to the degree of exactneſs that we think neceſſary. This method of finding a firſt near value, a, of the leaſt root of a propoſed equation that ſeems to have more than one real and affirmative root, is explained more at length in the third volume of the Collection of Mathematical Tracts, called *Scriptores Logarithmici*, in my Diſcourſe on the Reverſion of Infinite Serieſes publiſhed in that Volume, in pages 724, 725, 726, 727, &c, - - - to page 761; which pages will be reprinted in the ſubſequent part of this preſent Collection of Tracts. And, with this improvement of it in the caſe of equations that have, or ſeem to have, more than one real and poſitive root, I believe it may ſafely be affirmed that Mr. Raphſon's Method of Reſolving Affected Equations is the beſt general method of effecting that purpoſe in all equations above quadraticks that has hitherto been diſcovered.

End of the Obſervations on Mr. Raphſon's *Method of Reſolving Affected Equations by Approximation.*

An EXPLICATION *of* SIMON STEVIN's General Rule, *to Extract One Root out of any Possible Equation in Numbers, either Exactly, or very nearly True.*

By JOHN KERSEY.

BEING THE TENTH CHAPTER OF THE SECOND BOOK OF MR. KERSEY'S ELEMENTS OF ALGEBRA.

Article 1. EQUATIONS falling under any of the forms in the fourteenth and fifteenth chapters of the first book of these Elements, are capable (as hath there been shewn) of perfect resolutions in numbers; viz. the value of the root or roots sought in any of those equations may be found out and expressed exactly, either by some rational or irrational number or numbers; but the perfect resolution of all manner of compound equations in numbers, I have not found in any author: and since an exposition of the general method of Vieta, the rules of Huddenius and others to that purpose, would make a large treatise, and after all leave the curious Analyst dissatisfied, I shall not clogg these Elements with

a tedious

a tedious difcourfe upon thofe difficult rules, (which at the beft are exceeding tedious in operation, and fome of them uncertain too,) but rather purfue my firft defign, which was to explain fundamentals, and fuch rules as are certain and moft important in this profound art. However, I fhall lead the induftrious learner a few fteps farther in order to his underftanding the refolution of all manner of compound equations in numbers, and in this Chapter fhall explain Simon Stevin's General Rule, which, with the help of the rules in the following Eleventh Chapter, will difcover all the roots of any poffible equation in numbers, either exactly, if they be rational, or very nearly true, if irrational.

QUESTION I.

If - - - $aaa + 26a$ is $= 40188$, what is the number a?

RESOLUTION.

This equation not falling under any of the three forms in Sect. I. chap. xv. book 1. cannot be refolved by any of the canons in that chapter, and therefore according to Simon Stevin's general method I fearch out the number a by trials, thus, viz.

1. I

1. I ſuppoſe - - - - $a = 1$

Thence it follows that - - $aaa = 1$

And - - - - $26a = 26$

Therefore - - $aaa + 26a = 27$;

Which 27 ought to have been 40188, but it's too little; whereby I find that, by ſuppoſing a to be 1, I did not hit upon the true number a; and therefore I make another trial, in like manner as before, viz.

2. I ſuppoſe - - - - $a = 10$

Thence it follows that - - $aaa = 1000$

And - - - - $26a = 260$

Therefore - - $aaa + 26a = 1260$

Which 1260 being yet too little, I make a third trial, viz.

3. I ſuppoſe - - - $a = 100$

Thence it follows, that $aaa + 26a = 1{,}002{,}600$

Which 1,002,600 exceeds the juſt reſult, or abſolute number 40,188 in the latter part of the equation firſt propoſed, and therefore the true number a is leſs than 100; but the ſecond trial ſhews it to be greater than 10, and therefore the whole number which expreſſeth the exact, or, at leaſt, part of the value of a, muſt neceſſarily conſiſt of two characters, and conſequently the firſt (towards the left hand) muſt be one of theſe nine, 1, 2, 3, 4, 5, 6, 7, 8, 9. But, becauſe, by the ſecond inquiry,

10 was found too little, I now make trial with 2 for the firſt figure of the root a, viz.

4. I ſuppoſe - - - - $a = 20$
Thence - - $aaa + 26a = 8520$

Which reſult 8520 being yet leſs than the juſt reſult 40,188, I make trial again, viz.

5. I ſuppoſe - - - $a = 30$
Thence - - $aaa + 26a = 27780$

Which is yet too little; therefore,

6. I ſuppoſe - - - $a = 40$
Thence - - $aaa + 26a = 65,040$

Which 65,040 being greater than 40,188, it ſhews me that the true root or value of a is leſs than 40; but by the fifth trial it is greater than 30, and conſequently the firſt figure of the root is 3.

Now the ſecond character of the root muſt neceſſarily be one of theſe, viz. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; and becauſe it hath been diſcovered that the true value of the root a is greater than 30, the ſecond character cannot be 0: I therefore make trial with 1, and ſuppoſe $a = 31$, which proving too little, I make trial with 32, 33, 34, &c. ſeverally, in like manner as before, and at length I find 34 to be the true number a ſought, by which the equation propoſed may be expounded; for, if $a = 34$, then conſequently $aaa + 26a = 40,188$.

Art. 2.

Art. 2. But if after trials made (as before) the value of a, the root fought, happens to fall between two whole numbers that differ by unity; then trials are to be made with the leffer whole number increafed with $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c. until you have found the value of a in fome mixt number confifting of a whole number and fome certain tenth parts of an unit: But if the faid value of a happens not to be expreffed exactly by the faid leffer whole number increafed with certain tenth parts, then you are to make trials with the faid leffer whole number increafed with a decimal fraction having for it's numerator a number greater than 10, but lefs than 100; and for it's denominator 100, as with $\frac{11}{100}$, $\frac{12}{100}$, &c. and by proceeding in that manner you may find the exact value of the root a when it's fractional part is exactly equal to fome decimal fraction, or elfe approach infinitely near to the faid exact value when it is irrational or furd, as in this following:

QUESTION II.

If - - - $aaaa + 50a$ is $= 184{,}638.6801$; (or, $184{,}638\frac{6801}{10000}$;) what is the number a?

RESOLUTION.

First, I fuppofe $a = 1$; but this proving too little, I put $a = 10$; this alfo proving too little, I affume $a = 100$, which after trial I find to be greater than the true number a, and confequently the number a falls between 10 and

and 100; then, making trial with 20, I find it too little; but making trial with 30, I find this too great; and therefore the true root *a* falls between 20 and 30. Again, making trial with 21, I find it too great: but 20 was before found too little; therefore the true root *a* is between 20 and 21; then I make trial with 20.1, (that is, $20\frac{1}{10}$;) 20.2; 20.3, &c. and at length find 20.7 to be the true number *a* sought; for if $a = 20.7$, (that is, $20\frac{7}{10}$,) it will make $aaaa + 50a = 184{,}638.6801$ the equation proposed *.

But if 20.7 had proved too little, and 20.8 too great, then trials must have been made with 20.71, (that is, $20\frac{71}{100}$;) 20.72; 20.73, &c. In like manner, if 20.7 had been too little, but 20.71 (that is, $20\frac{71}{100}$) too great, then trials must have been made with 20.701, (that is, $20\frac{701}{1000}$;) 20.702; 20.703, &c. This will be partly exemplified in resolving the equation in this following—

QUESTION III.

If - - - $aaa + 20aa = 1954$, what is the number *a*?

Ans. - - $a = 8.308$, &c. found out by trials, as before.

* This method of proceeding is very convenient for finding the two first figures of the value of *a*; but then it will be expedient (for the sake of greater dispatch,) to have recourse to Mr. Raphson's method of approximation, which will usually double, or nearly double, the number of figures in the value of *a* at every operation. See his *Analysis Æquationum Universalis*.

Art. 3.

Art. 3. When the value of (a) the required root of an equation happens to be leſs than unity, then trial is to made with $\frac{1}{10}$; but if this prove too great, then with $\frac{1}{100}$, &c. Now ſuppoſe .1 (that is, $\frac{1}{10}$) be too great, but .01 (that is, $\frac{1}{100}$) too little; then trial muſt be made with .02 | .03 | .04 | &c. until you have found out the greateſt figure that muſt ſtand in the ſecond place of the decimal fraction expreſſing the root ſought; ſuppoſing then ſuch figure to be found 8, viz. that .08 (or $\frac{8}{100}$) is leſs, but .09 (or $\frac{9}{100}$) is greater than the root, trial muſt be made with .081, (that is, $\frac{81}{1000}$,) .082 | .083 | &c. as in this following—

QUESTION IV.

If $--aaa + 3240a$ is $= 269$, what is the number a?

Anſw. - a is $= .083$, &c. that is, $\frac{83}{1000}$, &c.

Art. 4. The preceeding examples may ſuffice to ſhew the uſe of this general method when all the terms of the unknown part of an equation are affirmative, (viz. when + is prefixed to each term,) in which caſe there is but one affirmative root; in the ſearch whereof by trials (as before) if the numbers aſſumed ſeverally for the value of the root ſought do aſcend, or become greater and greater, then the abſolute numbers reſulting from thoſe aſſumed values will likewiſe aſcend; and contrary, if the aſſumed roots do deſcend from a greater to a leſs, the reſults will likewiſe grow leſs and leſs: whence, by comparing an abſolute number reſulting from an aſſumed root with the juſt abſolute number of the equation propoſed, you may certainly

certainly know (if the ſaid reſult and juſt abſolute be not equal to one another) whether you are to take a number greater or leſs than that laſt before aſſumed.

But when the unknown part of an equation conſiſts of affirmative and negative terms mingled one with another, then the ſearch by trials will be more intricate and doubtful than before; for ſometimes it will be hard to diſcern whether a following aſſumed root muſt be taken greater or leſs than that which was taken next before. Moreover, a compound equation of this latter kind may happen to be ſuch, that it may be expounded by as many ſeveral affirmative roots as there be unities in the index of the higheſt unknown power, viz. a cubical equation may be ſo conſtituted that it ſhall have three different affirmative roots, a biquadratick equation four ſeveral roots; and ſo of higher equations, as will be ſhewn in the following Chap. 11. But, in what manner ſoever any poſſible equation is conſtituted in rational numbers, this general method will always find out one affirmative root, either exactly true, or at leaſt very near the truth; as will farther appear by the following queſtions:

QUESTION V.

If - - - - $aaa - 22aa + 157a = 360$, what is the number a?

RESOLUTION.

1. I ſuppoſe - - - - $a = 1$

Thence it follows that $aaa - 22aa + 157a = 136$

Which 136 is leſs than the juſt abſolute number 360, and therefore I make another trial, viz.

2. I ſuppoſe - - - - $a = 10$

Thence it follows that $aaa - 22aa + 157a = 370$

Which 370 exceeds the juſt abſolute number 360, and therefore I conclude there is one affirmative value of a (either rational or irrational) between 1 and 10, which value, after trials made with 2, 3, 4, 5, I find to be 5: this will conſtitute the equation propoſed; for if $a = 5$, then $aaa - 22aa + 157a$ will exactly make 360.

But there are two other roots or values of a, to wit, 8 and 9, each of which will likewiſe conſtitute the equation firſt propoſed; but how they are found out will be ſhewn in Sect. 9. of the following Chap. 11.

QUESTION VI.

If - - - - $3200a - aaa = 46,577$ (juſt,) what is the number a?

RESOLUTION.

1. I ſuppoſe - - $a = 1$

Thence - $3200a - aaa = 3199$ (leſs than juſt.)

2. I ſuppoſe - - $a = 10$

Thence - $3200a - aaa = 31000$ (leſs than juſt.)

3. I

3. I ſuppoſe - - $a = 100$

Thence - $3200a - aaa = -680,000$ (leſs than juſt.)

Now becauſe the ſecond reſult (or abſolute number) $+ 31000$ is affirmative, and the laſt reſult $- 680,000$ is negative, I make trials with numbers between 10 and 100 for the value of a; for, if the equation propoſed be poſſible, before the affirmative reſults fall off to negatives, there will be a root, or value of a, producing an affirmative reſult either exactly equal, or very near to the juſt reſult 46,577; therefore,

4. I ſuppoſe - - $a = 20$

Thence - $3200a - aaa = 56,000$ (greater than juſt.)

Now becauſe by taking 20 for the value of a, the reſult 56,000 exceeds the juſt reſult 46,577; but by taking 10 for a, the reſult 31000 happened to be leſs than the ſaid 46577; it ſhews that there is one affirmative root or value of a between 10 and 20, which root, after trials made with intermediate numbers (as in former examples) will be found 15.7, &c. Moreover, becauſe, by ſuppoſing $a = 20$, the reſult 56,000 happened to exceed the juſt reſult 46,577, but, by putting $a = 100$, the reſult $- 680000$ proved to be leſs than the ſame 46,577, it ſhews there is an affirmative value of a between 20 and 100, which value after trials made will be found 47: ſo that there are two affirmative roots or values of a found out, to wit, 15.7, &c. (or $15\frac{7}{10}$, &c.) and 47; the former of which will nearly, and the latter exactly, conſtitute the equation propoſed.

Art. 5.

Art. 5. Florimond de Beaune in the latter of two ſmall treatiſes printed in 1659, concerning the nature, conſtitution and limits of equations *, ſhews how to find-out limits within which the roots of all compound equations not aſcending above the biquadratick kind are confined; which limits, when they may be diſcovered without much trouble, and are not very wide aſunder, will help to leſſen the trials in the general method before delivered: As, in the laſt Example, where

The equation propoſed was $3200a - aaa = 46{,}577$

First, becauſe aaa muſt be ſubtracted from $3200a$ and leave a remainder equal to 46,577, it preſuppoſeth - - aaa to be $\sqsupset 3200a$

Therefore, by dividing each part by a, - - - - - $aa \sqsupset 3200$.

And by extracting the ſquare-root out of each part, it follows that - - - - - - - $a \sqsupset 56.5$, &c.

Again, from the equation propoſed, by tranſpoſition it is evident that - - - $3200a - 46{,}577 = aaa$

Whence it is alſo manifeſt - - $3200a \sqsubset 46{,}577$

And conſequently, by dividing each part by 3200, - - - $a \sqsubset 14.5$, &c.

* This tract of M. de Beaune is very clear and very uſeful. It is printed with Mr. Schooten's Comment on Des Cartes's Geometry.

Thus it is found that the value of a the root ſought is greater than 14.5, &c. but leſs than 56.5, &c. and therefore trials according to the general method aforeſaid need not be made with any numbers that are not within thoſe limits.

From the premiſes it is evident, that this general method finds not a perfect root of an equation, unleſs ſuch root be a whole number, or elſe a fraction exactly equal to ſome decimal fraction; or laſtly, a mixed number compoſed of a whole number and a perfect decimal fraction.

Note. When the co-efficients or known numbers multiplied into any of the unknown powers under the higheſt, (which muſt have no co-efficient but unity,) are vulgar (not decimal) fractions, or mixt numbers whoſe fractional parts are vulgar fractions; likewiſe, when the abſolute number that ſolely poſſeſſeth the latter part of the equation propoſed is a vulgar fraction, or a mixt number whoſe fractional part is a vulgar fraction; all thoſe vulgar fractions muſt be reduced to decimal fractions, or elſe the equation muſt be reduced to another equation in integers (by Sect. 7. in the following Chap. 11.) before you enter upon the reſolution by trials as aforeſaid.

End of the Tenth Chapter of the Second Book of Mr. Kerſey's *Elements of Algebra.*

A RE-

A REMARK

ON

AN ERROR IN MONSIEUR CLAIRAUT'S ELEMENTS OF ALGEBRA.

By FRANCIS MASERES, *Esq.*

CURSITOR BARON OF THE COURT OF EXCHEQUER.

A REMARK *on an Error in the Reaſoning of the late learned French Mathematician,* Monſieur CLAIRAUT, *in that Part of his* ELEMENTS OF ALGEBRA *in which he endeavours to prove the Rules of Multiplication laid down by Writers on Algebra concerning Negative Quantities.*

THERE is one writer of Algebra who has treated the ſubject with uncommon care and elegance, and, throughout the firſt part of his work, with great perſpicuity likewiſe. I mean Monſieur Clairaut of the French Academy of Sciences, in his *Elémens d'Algèbre* publiſhed at Paris in the year 1749. But even this writer has been betrayed into a moſt remarkable inſtance of falſe reaſoning, by his deſire of explaining to his readers that which in it's nature is not capable of being explained or underſtood, to wit, the propoſition "*that, if a negative quantity be multiplied by a negative quantity, the product will be an affirmative quantity.*" This propoſition he has attempted to demonſtrate in the 60th Section, or Article, of his book, page 73, in the following manner: He had already in Art. iii, page 4, explained the ſign — to ſignify the ſubtraction of the quantity to which it is prefixed from the quantity which immediately preceeds it, in theſe

words:

words: "*Prenant le caractere* —, *qui se prononce* moins, *pour faire ressouvenir que la quantité qu'il précéde doit être retranchée de celle qu'il suit.*" And in Articles xlii, xliii, xliv, xlv, xlvi, pages 48, 49, 50, 51, 52, 53, 54, he explains the manner of multiplying compound quantities, (which he calls *des quantités complexes ou polynomes, ou quantités composées de plusieurs termes,*) by each other, and exemplifies it by the multiplication of the compound quantity $2a^3c^2 - 5a^4b + 6a^5$ by the compound quantity $3ab^2 - 4bcd$. And thus far he reasons justly and clearly, using the sign — only in the sense of his first definition of it in Art. iii, page 4, as the sign of the subtraction of the quantity to which it is prefixed from that which goes before it, in which first part of his book he seems to suppose all along that the quantities to be subtracted are less than those from which they are to be subtracted. But then, when he has thus shewn clearly that, in the multiplication of compound quantities by compound quantities, the rule holds good, that — into — gives +, or that those members of the product of the two compound quantities which arise from the multiplication of any two members of the compound quantities themselves that are both marked with the sign —, will be marked with the sign +, he endeavours, in Art. lx, page 73, to extend this conclusion to the product of two independent, or separate, negative quantities, or quantities which are marked with the sign — without being preceeded by any other quantities from which they are subtracted. This lxth Article is in these words:

Pour nous assurer que la multiplication de — par — doit toûjours donner + au produit, voyons quelle lumière nous pouvons

pouvons tirer de la méthode générale des multiplications donnée dans l'Art. xlv. Suivant cette méthode on voit très clairement que le produit d'une quantité tellé que a — b *par une autre* c — d *doit être* ac — bc — ad + bd; *et on voit par conséquent en même temps qué le terme* bd, *qui est venu par la multiplication de* b *et de* d, *a le signe* +, *tandis que ses produisants* b *et* d *ont le signe* —. *Il ne reste donc plus qu'à sçavoir si, lorsque deux quantités negatives, telles que* b *et* d, *ne seront précedées d'aucune quantité positive, leur produit sera encore* + bd. *Or c'est ce dont il est facile de reconnoitre la vérité, puisque la méthode par laquelle on a découvert que le produit de* a — b *par* c — d *étoit* ac — bc — ad + bd, *ne spécifiant aucune grandeur particuliére ni à* a *ni à* c, *doit avoir encore lieu lorsque ces quantités sont égales à zéro. Or, en ce cas, le produit* ac — bc — ad + bd *se reduit à* + bd. *Donc* — b × — *d est* = + bd.

Now in these words there is an obvious fallacy. For, though *a* and *c* are not particular numbers, or quantities which have a fixt relation to *b* and *d*, but they may be of many different magnitudes with respect to *b* and *d*, yet they must always be supposed to be *greater* than *b* and *d* respectively. For otherwise it will not be possible to subtract *b* and *d* from them, so as to produce the quantities *a* — *b* and *c* — *d*; the author not having hitherto given us any other idea of the sign — but that of it's denoting the subtraction of the quantities to which it is prefixed from those which go before them; in order to which it is necessary, and is constantly supposed in all the foregoing part of the book, that the said quantities, to which the said sign is prefixed, should be less than those that go

before

before them. It follows therefore that a and c can never be ſuppoſed to be either equal to, or leſs than, b and d, reſpectively, though they may be of any magnitudes that are greater than b and d. It is not therefore poſſible for a and c to become equal to 0, as the author ſuppoſes; and conſequently the compound quantity $ac - bc - ad + bd$, (which is the product of the multiplication of $a - b$ and $c - d$ into each other) can never become equal to $+ bd$ by the vaniſhing, or annihilation, of it's three firſt members ac, bc, ad, as the author concludes from the ſaid falſe ſuppoſition that a and c may be taken equal to 0.

The author on this occaſion ſeems to have forgot his own definition of the ſign $-$, by which he made it to be a mark of the ſubtraction of the quantity to which it is prefixed from the quantity that goes before it; from which definition it is plain that the ſaid ſign always ſuppoſes the exiſtence of two different quantities, of which the one is to be ſubtracted from the other, and conſequently that it can have no meaning when applied to a ſingle quantity, as b or d, independently of ſome other and greater quantity, as a or c, from which it is to be ſubtracted. There cannot therefore exiſt any ſuch quantities as $- b$, or $- d$; and conſequently no propoſitions concerning them can be either true or falſe. Conſequently the propoſition which the author there endeavours to demonſtrate, to wit, "that $- b \times - d$ is equal to $+ bd$, when $- b$ and $- d$ are not preceeded by two greater quantities a and c from which they are ſubtracted, but are conſidered as ſingle and independent quantities," is ſo far from being

being *true* that it is not even *intelligible* according to the only idea of the ſign — which the author has given us in all the preceeding part of the book.

Such is the obſcurity and confuſion in which even this very able writer (who not only had a great deal of genius and ſagacity in the mathematical ſciences, but alſo an elegant taſte in his manner of treating them,) has involved himſelf by his attempt to explain the nature and properties of negative quantities. And other writers who have made the like attempt, have fallen into ſimilar abſurdities; amongſt whom we may reckon the celebrated Mr. Leonhard Euler, in his laſt work, called *The Elements of Algebra*, in two large volumes, octavo, which has been commended by ſome perſons as a very eaſy and clear treatiſe on that ſcience. But, upon examining it, I have found the firſt volume of it to abound with errors and difficulties ariſing from a moſt compleat adoption and very frequent uſe of this perplexing and abſurd doctrine of negative quantities, for which he ſeems to have an uncommon attachment.

After ſeeing that ſo able a mathematician as Monſieur *Clairaut* could be ſo far blinded and puzzled by this ſtrange doctrine of negative quantities (unneceſſarily introduced into the otherwiſe clear and ſimple ſcience of Algebra, or Univerſal Arithmetick,) as to reaſon ſo very weakly in ſupport of it as we have ſeen he did in the foregoing paſſage, it is ſurely high time for every true lover of this ſcience, who is zealous for the honour of it's purity and perſpicuity, to exclaim as the good Archbiſhop *Tillotſon* did with reſpect to the Athanaſian Creed, "*I wiſh we were fairly rid of it!*"

A GENERAL METHOD

OF

INVESTIGATING THE TWO, OR THREE, FIRST FIGURES OF THE LEAST ROOT OF AN EQUATION THAT HAS MORE THAN ONE REAL AND AFFIRMATIVE ROOT.

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A GENERAL METHOD *of investigating the Two, or Three, first Figures of the least Root of an Equation that has more than One real and affirmative Root.*

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Article 1. IN the foregoing Appendix to Dr. Halley's Tract on the Resolution of High Equations by Approximation, I have endeavoured to illustrate his method by a very full and distinct resolution of the three equations which he has chosen for examples of it, and to compare it with Mr. Raphson's method of resolving the like equations, (which also proceeds by repeated approximations to the true values of the roots sought,) by applying the latter method to the resolution of the same equations which had before been resolved by Dr. Halley's method; and the result of the comparison was, that Mr. Raphson's method appeared to me, for the most part, more convenient than Dr. Halley's. But in the application of both these methods of resolving these equations a difficulty may sometimes occur which it will be expedient to endeavour to remove. For, when the terms of an equation have many changes of their signs from + to — and from — to +, it may, perhaps, have many different roots, (I mean, real and affirmative roots,) of very different

ferent magnitudes from each other, and we may find ourselves at a loss to determine which of these several roots we ought to begin by, or go first in pursuit of. This difficulty, indeed, will not always occur in equations of this kind, because it will sometimes happen that the conditions of the problem from which the equation is derived, will point-out to us some limits to that root of the equation which is necessary to the solution of the problem, and which therefore ought to be the object of our investigation; and in these cases it is evident we must make use of these limits as our guides in our first conjectural approximation towards the value of the root that lies between, or near, them. But, when the conditions of the problems do not afford us such assistance, we may be a good deal puzzled to know which of the several roots which the equation possibly may have, we should endeavour to investigate first. And in these cases, I believe, it will often be found convenient to begin with the investigation of *the least* of the several different roots of the equation. The method of doing this, and the reasons for chusing to begin with the least root in these cases in preference to any of the other roots, will be the subject of the present discourse.

Art. 2. In many equations of all degrees, it is easy to perceive that they can have but one root, that is, but one real and affirmative root. For, as to negative roots, they are in truth the real and affirmative roots of other equations consisting of the same terms, or members, as the equations of which they are said to be the negative roots, but differently connected with each other by the signs

ſigns + and —, or by addition and ſubtraction: and, as to impoſſible roots, they are mere imaginary quantities, of which the mind can form no idea; and they are called roots of the equations they belong to, only becauſe, if they are ſubſtituted in the ſaid equations inſtead of x, they will make the left-hand ſides of the equations be equal, or, rather, ſeem to be equal, to their ſeveral abſolute terms reſpectively; or, in other words, becauſe, if they are ſquared, and cubed, and raiſed to the fourth, or fifth, or other higher, powers, and the ſaid ſquares, and cubes, and other powers of them, are multiplied into the coefficients of the ſame powers of x in the equations to which they belong, they will make the left-hand ſides of ſuch equations be equal to their ſeveral abſolute terms reſpectively. But it is impoſſible to ſquare, or cube, or multiply, a quantity that cannot exiſt, or a non-entity; and therefore all that is ſaid about theſe impoſſible roots is little better than ſtark nonſenſe, and tends only to darken and diſgrace the Science of Algebra. Nor can I conceive the ſmalleſt reaſon for ever mentioning theſe roots, or indeed negative roots, in books of Algebra, unleſs it be to ſupport the truth of a favourite poſition concerning equations that has been laid down by modern writers of Algebra, to wit, "that every Algebräick equation has as many roots as there are units in the index of the higheſt powers of x contained in it;" which poſition would, without the admiſſion of negative and impoſſible roots, be moſt eminently falſe. For many equations of all degrees have in truth only one root, or quantity, really exiſting, and of which we can form a clear conception, that, being ſubſtituted inſtead of x in the terms of the left-hand ſide of the equation, will make

the refult of them be equal to the abfolute term of the equation. And it is often eafy to perceive that thefe equations can have but one fuch root, though their terms may involve in them the cube, or the fourth power, or the fifth power, or any higher powers, of the unknown quantity x; of which I will mention two, or three, of the moft remarkable inftances.

Art. 3. In the firft place, when all the terms on the left-hand fide of the equation, or that involve in them any powers of the unknown quantity x, are added to each other, or connected together by the fign $+$, it is obvious that fuch an equation can have but one root.

Thus, for example, if the equation be $x^3 + 36xx + 44x = 1$, it is evident that x can have but one value. For, if x were to increafe from it's firft value, or the value which it has when the compound quantity $x^3 + 36x^2 + 44x$ becomes firft equal to the abfolute term 1, to any greater magnitude, each of the three quantities x^3, $36x^2$, and $44x$ would, at the fame time, increafe continually, or without ever decreafing, from it's firft value to a greater quantity, and confequently their fum $x^3 + 36x^2 + 44x$ would alfo increafe continually, or without ever decreafing, from it's firft value, or 1, to a greater quantity, and therefore could never become a fecond time equal to 1.

And, in like manner, the biquadratick equation $x^4 + 80x^3 + 1998x^2 + 14{,}937x = 5000$ can have but one root. For, if x were to increafe from it's firft value, or the

the value it has when the compound quantity $x^4 + 80x^3 + 1998x^2 + 14{,}937x$ is firft equal to 5000, to any greater quantity, each of the four terms x^4, $80x^3$, $1998x^2$, and $14{,}937x$ would increafe at the fame time continually, or without ever decreafing, from it's firft value to fome greater quantity; and confequently their fum, or the compound quantity $x^4 + 80x^3 + 1998x^2 + 14{,}937x$, would alfo at the fame time increafe continually, or without ever decreafing, and therefore could never become a fecond time equal to the abfolute term 5000.

And the like would be evident in any other equation whatfoever, in which all the terms on the left-hand fide of the equation, or all the terms that involve in them the powers of x, were added to each other, or connected together by the fign $+$. Therefore every fuch equation can have but one root.

Art. 4. Secondly, when the term that involves the higheft power of x is greater than the fum of all the other terms on the left-hand fide of the equation, and the excefs of the faid term above the faid fum of all the other terms on that fide of the equation is equal to the abfolute term of the equation, (as, for example, if x^3 is greater than $36xx + 44x$, and $x^3 - 36x^2 - 44x$ is equal to 1; or, if x^4 is greater than $80x^3 + 1998xx + 14{,}937x$, and $x^4 - 80x^3 - 1998xx - 14{,}937x$ is equal to 5000;) the equation can have but one root. And this root will be greater than the value of x that would refult from a fuppofition that the abfolute term of the equation was equal to 0.

Thus,

Thus, for example, the root of the cubick equation $x^3 - 36x^2 - 44x = 1$ muſt be greater than the root of the equation $x^3 - 36x^2 - 44x = 0$, or than the root of the quadratick equation $xx - 36x - 44 = 0$, or $xx - 36x = 44$; and the root of the bi-quadratick equation $x^4 - 80x^3 - 1998x^2 - 14,937x = 5090$ will be greater than the root of the equation $x^4 - 80x^3 - 1998xx - 14,937x = 0$, or than the root of the cubick equation $x^3 - 80xx - 1998x - 14,937 = 0$, or $x^3 - 80xx - 1998x = 14,937$.

Art. 5. The truth of theſe poſitions will be evident from conſidering the manner in which the ſeveral terms involving the powers of x in any of theſe equations will increaſe while x increaſes from 0 *ad infinitum.*

For, when x is very ſmall, all the other terms on the left-hand ſide of the equation, which will involve x^2, x^3, x^4, &c, will be much ſmaller than x, the ſaid powers of x being continued proportionals to 1 and x. And x may be taken of ſo ſmall a magnitude that the proportion of x to x^2 ſhall be greater than any ratio of majority that ſhall have been aſſigned. But, as x increaſes, the other terms, (which involve x^2, x^3, x^4, &c,) will increaſe faſter than x; and the ratio of x to x^2 will decreaſe continually, from having been a very great ratio of majority, till it becomes a ratio of equality when x is equal to 1. And at this time all the powers of x, to wit, x^2, x^3, x^4, x^5, &c, will be equal to each other and to 1. And, when x increaſes further from 1 to a greater magnitude, x^4 will be greater than x, and likewiſe than x^2 and x^3, and it's increment in-

in any given portion of time will be greater than the contemporary increment of x, and likewise than the contemporary increments of either of the other powers of x that are lower than itself, to wit, x^2 and x^3. And, by this means, x^4, from having been less than either $14,937x$, or $1998xx$, or $80x^3$, will gain ground upon them, and, at some one point of time, become equal to the sum of all the three together; and at this time the compound quantity $x^4 - 80x^3 - 1998x^2 - 14,937x$ will be equal to 0, and the value of x will be the root of the biquadratick equation $x^4 - 80x^3 - 1998x^2 - 14,937x = 0$, or the root of the cubick equation $x^3 - 80x^2 - 1998x - 14,937 = 0$, or $x^3 - 80x^2 - 1998x = 14,937$. And, when x increases further from this value *ad infinitum*, the increment of x^4 (which has already been greater than the sum of the contemporary increments of $80x^3$, and $1998x^2$, and $14,937x$, so as to enable x^4, from having been at first less than either of the three quantities, to become equal to all the three put together,) will continue to be greater than the sum of the three contemporary increments of those three quantities, $80x^3$, $1998x^2$, and $14,937x$, and in a continually-increasing ratio of majority. And consequently the compound quantity $x^4 - 80x^3 - 1998x^2 - 14,937x$, or the excess of the single quantity x^4 above the sum of the three quantities $80x^3$, $1998x^2$, and $14,937x$, will increase continually, or without ever decreasing, from 0 *ad infinitum*, while x increases from being equal to the root of the cubick equation $x^3 - 80x^2 - 1998x = 14,937$ *ad infinitum*. Therefore the said compound quantity $x^4 - 80x^3 - 1998x^2 - 14,937x$ will, at

fome one inftant or time during it's faid increafe from 0 *ad infinitum*, become equal to the abfolute term 5000, or to any other given quantity whatfoever; but it will be equal to fuch quantity only once during it's faid increafe from 0 *ad infinitum.* And confequently the equation $x^4 - 80x^3 - 1998x^2 - 14,937x = 5000$ can have but one root, and that root muft be greater than the root of the biquadratick equation $x^4 - 80x^3 - 1998x^2 - 14,937x = 0$, or than the root of the cubick equation $x^3 - 80x^2 - 1998x = 14,937$. Q. E. D.

And it is evident that the fame things will take place in any other equation whatfoever, in which the term that involves the higheft power of x is greater than the fum of all the other terms of the equation that involve the other, or inferiour, powers of x, and it's excefs above the faid fum is equal to the abfolute term, or in which all the terms on the left-hand fide of the equation, except that which involves the higheft power of x, are fubtracted from that term, and confequently marked with the fign —.

Art. 6. In the third place, when the term that involves the higheft power of x in the equation is added to the next term, or term involving the next power of x, or to the two next, or the three next terms, or any greater number of the terms next following it, and their fum is greater than the fum of all the remaining terms on the left-hand fide of the equation, or than all the other terms that involve the lower powers of x, and the latter fum

fum is fubtracted from the former, the equation can have but one root.

Thus, for example, the equation $x^4 + 80x^3 - 1998x^2 - 14{,}937x = 5000$, and the equation $x^4 + 80x^3 + 1998x^2 - 14{,}937x = 5000$, can have but one root a-piece. And all the following equations will have but one root a piece, to wit,

$$x^7 + 3x^6 - 5x^5 - 7x^4 - 9x^3 - 11x^2 - 13x = 15,$$
and $$x^7 + 3x^6 + 5x^5 - 7x^4 - 9x^3 - 11x^2 - 13x = 15,$$
and $$x^7 + 3x^6 + 5x^5 + 7x^4 - 9x^3 - 11x^2 - 13x = 15,$$
and $$x^7 + 3x^6 + 5x^5 + 7x^4 + 9x^3 - 11x^2 - 13x = 15,$$
and $$x^7 + 3x^6 + 5x^5 + 7x^4 + 9x^3 + 11x^2 - 13x = 15.$$

This will eafily appear from reafonings concerning the manner in which the feveral terms involving the powers of x increafe, and in which the excefs of the fum of the terms involving the higher powers of x above the fum of the terms involving the lower powers of x increafes, exactly fimilar to the reafonings ufed in the foregoing article. And therefore I do not think it neceffary to be more particular in the proof of it.

Art. 7. And there are fome other cafes in which we may be fure before-hand that an equation has only one real and affirmative root, though the figns of the terms on the left-hand fide of the equation fhould vary from + to — and from — to + two or three times, or oftener. But thefe cafes are more difficult to diftinguifh and afcertain than the three cafes before-mentioned. And therefore I fhall fay nothing further concerning them.

Art. 8. In all the cafes above-defcribed, and in all others where we know before-hand that the equation that is to be refolved has only one real and affirmative root, the beft way of beginning the inveftigation of the value of it's root, in order to approximate further to it afterwards by Mr. Raphfon's method above-mentioned, (if the conditions of the problem from which the equation is derived do not themfelves point out certain limits of the magnitude of the faid root, which will afford us a fufficiently near value of it to begin our approximation with, which they often will be found to do,) will be that invented by Stevinus, a mathematician of Bruges in the Spanifh Netherlands, who flourifhed in the beginning of the laft century, and died in the year 1633. This method is as follows:

Since the propofed equation can have but one real and affirmative root, it is evident that the whole compound quantity which forms the left-hand fide of the equation will increafe continually, or without ever decreafing, at the fame time that x increafes. Therefore, if we fubftitute any particular value for x in the faid compound quantity, and we find that the value of the faid compound quantity arifing from fuch fubftitution is greater than the abfolute term of the propofed equation, we may conclude with certainty that the faid fubftituted value of x is greater than the value of x in the propofed equation, or than the root of the faid equation: and, if the value of the faid compound quantity arifing from fuch fubftitution is lefs than the abfolute term of the equation, we may

may conclude with certainty that the ſaid ſubſtituted value of x is leſs than the root of the ſaid equation. Let us therefore firſt ſubſtitute 1 inſtead of x in the ſaid compound quantity, and try what the reſult will be; and, if the reſult is much leſs than the abſolute term of the equation, and conſequently 1 is much leſs than the true root of the equation, let us in the next place ſuppoſe x to be equal to 10, or to 9, or to 8, or to 7, or ſuch other number conſiſting of one or at moſt two figures, as we conjecture to be neareſt to the true value of x in the propoſed equation; and let this ſecond conjectural value of x be ſubſtituted inſtead of x in the ſaid compound quantity which forms the left-hand ſide of the equation. And thus, by a few eaſy trials, we ſhall ſoon find a value of x that will be true to at leaſt one figure, and without much difficulty we may find a value of it that ſhall be true in the two firſt places of figures; and we ſhall know at the ſame time whether the ſaid value is greater or leſs than the true value of x in the propoſed equation, becauſe it will be greater, if the reſult of the ſubſtitution of it in the aforeſaid compound quantity, which forms the left-hand ſide of the equation, is greater than the abſolute term of the equation; and it will be leſs, if the ſaid reſult is leſs than the ſaid abſolute term. And from this firſt near value of x, ſo obtained by theſe eaſy trials, or ſubſtitutions, we may begin our approximation to a more exact value of x in the manner deſcribed by Mr. Raphſon, and which has been exemplified above in the Appendix to Dr. Halley's Tract. Nothing more therefore need be ſaid on the re-

solution of equations of this kind, which we know beforehand to have but one real and affirmative root.

The foregoing method of finding the two first figures of the root of an equation of this kind is very fully illustrated by judicious examples by the learned Mr. John Kersey, in his excellent Treatise of Algebra, published in the latter part of the last century, in the years 1673 and 1674, in two volumes, folio, than which, I believe, there is not a better Treatise on that subject to be met with *. See Vol. I, Book II, Chapter 10, pages 265, 266, 267; which 10th chapter is also reprinted above in the foregoing part of this present work, pages 325, 326, &c - - - 338.

Art. 9. But, when we meet with a cubick, or a biquadratick, equation, or an equation of any higher order, in which there are two, or more, changes of the signs + and — that are prefixed to the terms that involve the powers of x, and in which consequently there are, or, at least, may be, for aught we know to the contrary, two or more real and affirmative roots, or different values of x,

* The learned Dr. John Wallis, in the Preface to the Latin Edition of his Algebra, in the 2nd volume of the Collection of all his Works published at Oxford in the year 1693, speaks of Mr. Kersey's Algebra in these words: *Suaserim ut Lector consulat ex nostris (præter alios,) Kersæum nostrum; qui duobus voluminibus integram Algebræ tractationem exhibuit, fusè quidem at perspicuè traditam. Quo nemo feliciùs quæstiones Diophantæas elucidavit.*

which

which, being ſubſtituted inſtead of x in the terms that form the left-hand ſide of the equation, will make the reſult of them equal to the right-hand ſide of it, or it's abſolute, or known, term, it is often difficult to know which root we had beſt begin to inveſtigate. And, if we make conjectures concerning the middle values of x, or if we make conjectures concerning it's values in general;—and the quantities we pitch upon for one of it's values are greater than it's leaſt value and nearer to one of it's middle values;—and we ſubſtitute one of theſe quantities inſtead of x in the compound quantity that forms the left-hand ſide of the equation;—and we find, after ſuch ſubſtitution, that the value of the ſaid compound quantity thence ariſing is greater than the abſolute term of the equation;—we ſhall not be able to conclude from thence that the ſaid conjectural value of x is greater than it's true value; nor, *vice verſâ*, if the value of the ſaid compound quantity reſulting from ſuch ſubſtitution is leſs than the abſolute term of the equation, ſhall we be able to conclude that the ſaid conjectural value of x is leſs than it's true value. For, in theſe equations which have more than one real and affirmative root, the compound quantity which forms the left-hand ſide of the equation will ſometimes decreaſe at the ſame time that x increaſes, as may be ſeen in Chap. X, pages 71, 72, &c - - - 90, of my Diſſertation on the Uſe of the Negative Sign in Algebra. In theſe equations therefore, which have, or ſeem to have, more than one real and affirmative root, there is often a good deal of difficulty in knowing how to ſet about the reſolution of them by Mr. Raphſon's method of approximation. And, I believe, it

will usually be found expedient in these cases to confine ourselves to the investigation of the least root, or value of x, in the equation. For, from the foregoing reasonings concerning the increase of x from 0 *ad infinitum*, and the contemporary increase of each of the terms in the equation that involves x, and the contemporary increase, or decrease, of the whole compound quantity that forms the left-hand side of the equation, it is evident that in the first part of the increase of x from 0 *ad infinitum*, or while it increases from 0 till it becomes equal to the least root of the proposed equation, the said compound quantity will increase with it continually, or without ever decreasing, though afterwards, when x is greater than the said least root, the said compound quantity will, if the equation has more than one real and affirmative root, sometimes decrease while x increases. This circumstance will enable us to make our conjectural approaches to the value of the said least root of the equation with more facility and perspicuity than to the other roots of it, and will enable us to conclude with certainty whether the value we have either assumed by a conjecture, or found by any previous method of investigation, for the said least root of the equation, or least value of x, is greater, or less, than it's true value. And on this account I am much inclined to think that the best way of resolving an equation of this kind is to begin with the investigation of it's least root. And the method I would recommend for this purpose is a very simple and easy one, being grounded on the following obvious proposition:

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Art. 10. If there be any equation involving ſeveral different powers of the unknown quantity x that are connected with each other both by addition and ſubtraction, or by both the ſigns + and —, ſuch as the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$; and there be another equation that has the ſame abſolute term r as the former equation, but has only ſo many terms in the left-hand ſide of it as there are terms in the left-hand ſide of the former equation connected together by the ſign +, which are four, to wit, x^7, lx^5, nx^3, and qx; and, if the unknown quantity contained in this ſecond equation be called y, and the powers of y involved in theſe terms of the ſecond equation are the ſame as the powers of x involved in the ſaid terms of the firſt equation which are connected together by the ſign +; and, if the co-efficients of the ſaid powers of y in the ſecond equation are the ſame as the co-efficients of the ſame powers of x in the firſt equation; and, laſtly, if all theſe terms of the ſaid ſecond equation involving the ſaid powers of y be added to each other, or connected together by the ſign +: So that the ſaid ſecond equation ſhall be $y^7 + ly^5 + ny^3 + qy = r$:—Then will the value of y, or the only root of the ſecond equation $y^7 + ly^5 + ny^3 + qy = r$, be leſs than the leaſt root of the firſt equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

DE-

DEMONSTRATION.

If there be two equations $x^7 + lx^5 + nx^3 + qx = r + D$, and $y^7 + ly^5 + ny^3 + qy = r$, in both which the terms which form the left-hand ſides of the equations are connected together by the ſign $+$, and involve the very ſame powers of the two unknown quantities x and y multiplied into the very ſame co-efficients 1, l, n, and q, but the abſolute term of the former equation exceeds the abſolute term of the latter equation by any given quantity called D, it is evident, from art. 3, that x, or the only root of the former equation $x^7 + lx^5 + nx^3 + qx = r + D$, will be greater than y, or the only root of the latter equation $y^7 + ly^5 + ny^3 + qy = r$.

Now let the three terms kx^6, mx^4, and px^2 in the equation $x^7 - kx^6 + lz^5 - mx^4 + nx^3 - px^2 + qx = r$, which are marked with the ſign $-$, be added to both ſides of the equation. Then, it is evident, we ſhall have $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$; which will have the very ſame roots as the former equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, becauſe it is only the ſame equation under another form. Therefore the leaſt root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$ will be alſo the leaſt root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

Let the leaſt root of this equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$ be called a, and it's other roots,

roots, if it has any other roots, be denoted by the letters b, c, d, &c.

Then it is evident that the trinomial quantity $kx^6 + mx^4 + px^2$ will have as many different values as there are different values of x, or different roots of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, or of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, and will be equal either to the trinomial quantity $ka^6 + ma^4 + pa^2$, or to the trinomial quantity $kb^6 + mb^4 + pb^2$, or to the trinomial quantity $kc^6 + mc^4 + pc^2$, or to the trinomial quantity $kd^6 + md^4 + pd^2$, &c; and the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$ will reprefent as many different equations, having one root a-piece, as it has different roots, to wit, 1ſt, the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$, which will have only one root, to wit, a; and, 2dly, the equation $x^7 + lx^5 + nx^3 + qx = r + kb^6 + mb^4 + pb^2$, which will have only one root, to wit, b; and, in like manner, the equations $x^7 + lx^5 + nx^3 + qx = r + kc^6 + mc^4 + pc^2$, and $x^7 + lx^5 + nx^3 + qx = r + kd^6 + md^4 + pd^2$, &c, which have each only one root, to wit, c, and d, &c, reſpectively. Therefore, if we ſubſtitute, inſtead of the trinomial quantity $kx^6 + mx^4 + px^2$, any one of it's particular values, $ka^6 + ma^4 + pa^2$, or $kb^6 + mb^4 + pb^2$, or $kc^6 + mc^4 + pc^2$, or $kd^6 + md^4 + pd^2$, &c, in the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, the equation thence ariſing will not have ſeveral different roots, as the ſaid equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$ had, but only one root, to wit, that root of the ſaid general equation which was ſubſtituted inſtead of x in the trinomial quantity $kx^6 + mx^4 + px^2$.

Let us therefore ſuppoſe the trinomial quantity $kx^6 + mx^4 + px^2$ to be equal to the trinomial quantity $ka^6 + ma^4 + pa^2$, which is it's leaſt value; and let $ka^6 + ma^4 + pa^2$ be denoted by the letter D.

Then will $x^7 + lx^5 + nx^3 + qx$ be $= r + D$; and this equation will have but one root, which will be equal to a, or the leaſt root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

But it has been obſerved above, that the root of the equation $x^7 + lx^5 + nx^3 + qx = r + D$ will be greater than the root of the equation $y^7 + ly^5 + ny^3 + qy = r$.

Therefore the leaſt root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, will be greater than the root of the equation $y^7 + ly^5 + ny^3 + qx = r$; or the root of the equation $y^7 + ly^5 + ny^3 + qy = r$ will be leſs than the leaſt root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

Q. E. D.

The Application of the foregoing Lemma to the Inveſtigation of a near Value of the Leaſt Root of an Equation that has more than One real and affirmative Root.

Art. 11. This Lemma being perfectly underſtood, I would propoſe to begin the reſolution of any equation of the

the foregoing form $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, (or in which there are two, or more changes of the ſigns, + and —, of the terms that involve the unknown quantity x,) by finding the ſole root of the equation $y^7 + ly^5 + ny^3 + qy = r$, which is derived from the former equation by dropping all the terms that have the ſign — prefixed to them, and by ſubſtituting y inſtead of x in the remaining terms. This root I would find in the manner preſcribed by Stevinus and Kerſey, to about two places of figures, and would denote it by the ſmall Greek letter α, to diſtinguiſh it from a, or the true value of the leaſt root of the original equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, or $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the ſole root of the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$; than which, by the foregoing Lemma, it would neceſſarily be leſs. And this quantity α I would conſider as the firſt near value of a, or the leaſt root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, which we are ſeeking.

Art. 12. In the next place I would ſubſtitute α inſtead of x in the compound quantity $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx$, (which forms the left-hand ſide of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$,) in order to find how far the value of the ſaid compound quantity reſulting from this ſubſtitution, or how far the compound quantity $\alpha^7 - k\alpha^6 + l\alpha^5 - m\alpha^4 + n\alpha^3 - p\alpha^2 + q\alpha$, (which will neceſſarily be leſs than the abſolute term r,) will fall ſhort of the abſolute term r. And, if I found that the ſaid compound quantity

tity $\alpha^7 - k\alpha^6 + l\alpha^5 - m\alpha^4 + n\alpha^3 - p\alpha^2 + q\alpha$ was considerably lefs than the abfolute term r, as, for example, not more than two third parts of it, I would increafe the abfolute term r of the equation $y^7 + ly^5 + ny^3 + qy = r$ by the addition of the three terms $k\alpha^6$, $m\alpha^4$, and $p\alpha^2$, and would feek the root of the new equation thence arifing, to wit, the equation $y^7 + ly^5 + ny^3 + qy = r + k\alpha^6 + m\alpha^4 + q\alpha$, to two places of figures, by the method of Stevinus and Kerfey, in the fame manner as we had before found the root of the equation $y^7 + ly^5 + ny^3 + qy = r$; and this root I would denote by the fmall Greek letter β.

Art. 13. This quantity β, being the fole root of the equation $y^7 + ly^5 + ny^3 + qy = r + k\alpha^6 + m\alpha^4 + p\alpha^2$, would be greater than α, or the fole root of the equation $y^7 + ly^5 + ny^3 + qy = r$, becaufe the abfolute term $r + k\alpha^6 + m\alpha^4 + p\alpha^2$ of the former equation is greater than r, or the abfolute term of the latter equation. And it will be lefs than a, or the fole root of the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$, becaufe the abfolute term $r + k\alpha^6 + m\alpha^4 + p\alpha^2$ is lefs than the abfolute term $r + ka^6 + ma^4 + pa^2$. Therefore it will be of an intermediate magnitude between α and a, and will approach nearer to a than the quantity α did, and confequently may juftly be called the fecond near value of a, or of the fole root of the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$, or of the leaft root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the original equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

Art. 14.

Art. 14. Having thus found these two near values, α and β, of a, or the least root of the original equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, by resolving the two equations $y^7 + ly^5 + ny^3 + qy = r$ and $y^7 + ly^5 + ny^3 + qy = r + ka^6 + ma^4 + pa^2$ in the manner prescribed by Stevinus and Kersey, I would denote the difference by which the second of these near values of a, to wit, β, falls short of a, or the least root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, by the letter z, and would substitute $\beta + z$ instead of x in the said equation, omitting all the terms that would involve in them any powers of z except the simple power of it, or z itself, agreeably to the directions of Mr. Raphson in his *Analysis Æquationum Universalis* above-mentioned, and would resolve the new equation resulting from such substitution, as a simple equation; by which we should obtain a near value of z, which, being added to β, would give us the value of $\beta + z$, or a third near value of a, or of the sole root of the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$, or of the least root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$. And then we might go on to find a fourth, and a fifth, and other following near values of a, or the said least root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$ by Mr. Raphson's method of approximation, in the manner exemplified above in the Appendix to Dr. Halley's Tract, till we had found it to as great degree of exactness as we desired.

This method of proceeding will be better understood by

by applying it to the refolution of two or three particular equations.

An Example of the Refolution of a Cubick Equation by the Method above-defcribed.

Art. 15. Let it be required to find the leaft root of the cubick equation $x^3 - 39x^2 + 479x = 1881$.

This equation has three real and affirmative roots, of which we will now endeavour to find the leaft in the manner juft now defcribed.

In the firft place, we muft feek by the method of Stevinus and Kerfey, the fole root of the equation $y^3 + 479y = 1881$; which will be lefs than the leaft root of the propofed equation $x^3 - 39x^2 + 479x = 1881$, or $x^3 + 479x = 1881 + 39x^2$, or (if we put a for the leaft value of x in this equation,) than the fole root of the equation $x^3 + 479x = 1881 + 39a^2$. Now the fole root of the equation $y^3 + 479y = 1881$ may be found by the method prefcribed by Stevinus and Kerfey, by proceeding as follows:

If y is $= 1$, the compound quantity $y^3 + 479y$ will be $(= 1 + 479) = 480$; which is lefs than 1881. Therefore 1 will be lefs than the true value of y.

Secondly,

Secondly, if y is $= 10$, the compound quantity $y^3 + 479y$ will be $(= 1000 + 4790) = 5790$; which is greater than 1881. Therefore 10 will be greater than the true value of y.

Thirdly, ſince y is greater than 1, but leſs than 10, let us ſuppoſe it to be equal to 5. Then we ſhall have $y^3 + 479y$ $(= 125 + 479 \times 5 = 125 + 2395) = 2520$; which is greater than 1881. Therefore 5 will be greater than the true value of y.

Fourthly, ſince y is leſs than 5, let us ſuppoſe it to be $= 4$. Then we ſhall have $y^3 + 479y$ $(= 64 + 479 \times 4 = 64 + 1916) = 1980$; which is greater than 1881. Therefore 4 is greater than the true value of y. But the difference cannot be great.

Fifthly, ſince y is leſs than 4, but pretty nearly equal to it, let us ſuppoſe it to be $= 3.8$. Then we ſhall have $y^3 + 479y$ $(= \overline{3.8}|^3 + 479 \times 3.8 = 54.872 + 1820.2)$ $= 1875.072$; which is a little leſs than 1881. Therefore 3.8 will be a little leſs than the true value of y in the equation $y^3 + 479y = 1881$. This value of y, or of the ſole root of the cubick equation $y^3 + 479y = 1881$, is near enough to it's true value for our preſent purpoſe. We muſt therefore conſider 3.8 as the firſt near value of a, or of the leaſt root of the original equation $x^3 - 39xx + 479x = 1881$, or of the equation $x^3 + 479x = 1881 + 39xx$, or of the ſole root of the equation $x^3 + 479x = 1881 + 39a^2$. This firſt near value of the ſaid root, or a, we will denote by the ſmall Greek letter α.

Art. 16. We muſt now proceed to find the value of the ſole root of the equation $y^3 + 479y = 1881 + 39a^2$, (or $y^3 + 479y = 1881 + 39 \times \overline{3.8}|^2$, or $y^3 + 479y = 1881 + 39 \times 14.44$, or $y^3 + 479y = 1881 + 563.16$,) or $y^3 + 479y = 2444.16$ by the method of Stevinus and Kerſey.

Now we have ſeen already that, if y is $= 5$, we ſhall have $y^3 + 479y = 2520$; which is greater than 2444.16. Therefore 5 will be greater than the true value of y. But the difference between them will be but ſmall; ſo that there will be no occaſion to determine the value of y in this equation to any greater degree of exactneſs. We may therefore conſider 5 as the ſecond near value of a, or of the leaſt root of the original equation $x^3 - 39x^2 + 479x = 1881$, or of the equation $x^3 + 479x = 1881 + 39x^2$, or of the ſole root of the equation $x^3 + 479x = 1881 + 39a^2$. This ſecond near value of the ſaid root, or a, we will denote by the ſmall Greek letter β.

Art. 17. Having thus obtained 5 for the ſecond near value of the leaſt root of the original equation $x^3 - 39xx + 479x = 1881$, we may proceed to inveſtigate the ſaid root to a greater degree of exactneſs by Mr. Raphſon's method of approximation, by putting z for the difference by which 5 falls ſhort of the true value of the ſaid leaſt root, and ſubſtituting $5 + z$ inſtead of x in the ſaid equation, with an omiſſion of all the terms which involve any powers of z, except the ſimple power of it, or z itſelf. This may be done as follows:

9 Since

Since x is $= 5 + z$, we shall have

$$xx\ (= \overline{5 + z}|^2 = \overline{5}|^2 + 2 \times 5z + \&c) = 25 + 10z + \&c,$$

and $x^3\ (= \overline{5 + z}|^3 = \overline{5}|^3 + 3 \times \overline{5}|^2 \times z + \&c = 125 + 3 \times 25 \times z + \&c) = 125 + 75z + \&c,$

and $39xx\ (= 39 \times \overline{25 + 10z + \&c} = 39 \times 25 + 39 \times 10z + \&c) = 975 + 390 \times z + \&c,$

and $479x\ (= 479 \times \overline{5 + z} = 479 \times 5 + 479 \times z) = 2395 + 479z,$

and consequently $x^3 - 39x^2 + 479x =$

$$\left\{\begin{array}{rrrl} & 125 & + \ 75z & + \ \&c \\ - & 975 & - \ 390z & - \ \&c \\ + & 2395 & + \ 479z & \end{array}\right\}$$

$$= \left\{\begin{array}{rrrl} & 2520 & + \ 554z & + \ \&c \\ - & 975 & - \ 390z & - \ \&c \end{array}\right\}$$

$$= 1545 + 164z \ \&c.$$

But $x^3 - 39x^2 + 479x$ is $= 1881$.

Therefore $1545 + 164z$ &c will also be $= 1881$. And consequently $164z$ will be $(= 1881 - 1545) = 336$, and z will be $= \frac{336}{164} = 2.0$ &c.

Therefore x, or $5 + z$, will be $= 5 + 2.0$ &c $= 7.0$ &c, or will be greater than 7.

Art. 18. Now let 7 be ſubſtituted inſtead of x in the compound quantity $x^3 - 39x^2 + 479x$, in order to diſcover whether the reſult will be greater, or leſs, than 1881, or the abſolute term of the equation $x^3 - 39x^2 + 47x = 1881$; and conſequently whether 7 is greater, or leſs, than the true value of the leaſt root of the ſaid equation. By this ſubſtitution we ſhall have $x^3 - 39x^2 + 479x$ $(= \overline{7}|^3 - 39 \times \overline{7}|^2 + 479 \times 7 = 343 - 39 \times 49 + 3353 = 3696 - 1911) = 1785$; which is leſs than 1881. Therefore 7 is leſs than the true value of the leaſt root of the propoſed equation $x^3 - 39x^2 + 479x = 1881$.

Art. 19. Let us therefore ſuppoſe x to be $= 7 + v$, and ſubſtitute $7 + v$ inſtead of x in the equation $x^3 - 39x^2 + 479x = 1881$. And we ſhall have

$$xx\ (= \overline{7 + v}|^2) = \overline{7}|^2 + 2 \times 7 \times v + \&c = 49 + 14v + \&c,$$

and x^3 $(= \overline{7 + v}|^3 = \overline{7}|^3 + 3 \times \overline{7}|^2 \times v + \&c = 343 + 3 \times 49 \times v + \&c) = 343 + 147 \times v + \&c,$

and $39xx$ $(= 39 \times \overline{49 + 14v + \&c} = 39 \times 49 + 39 \times 14v + \&c) = 1911 + 546 \times v + \&c,$

and $479x$ $(= 479 \times \overline{7 + v} = 479 \times 7 + 479 \times v) = 3353 + 479v,$

and conſequently $x^3 - 39xx + 479x =$

343

$$\left\{\begin{array}{r} 343 + 147 \times v + \&c \\ -\ 1911 - 546 \times v - \&c \\ +\ 3353 + 479 \times v + \&c \end{array}\right\}$$

$$= \left\{\begin{array}{r} 3696 + 626 \times v + \&c \\ -\ 1911 - 546 \times v - \&c \end{array}\right\}$$

$$= 1785 + 80 \times v\ \&c.$$

But $x^3 - 39xx + 479x$ is $= 1881$.

Therefore $1785 + 80 \times v$ &c will alſo be $= 1881$. And conſequently $80 \times v$ will be ($= 1881 - 1785$) $= 96$, and v will be $= \frac{96}{80} = 1.2$. Therefore x, or $7 + v$, will be $(= 7 + 1.2) = 8.2$.

Art. 20. Now let 8.2 be ſubſtituted inſtead of x in the compound quantity $x^3 - 39x^2 + 479x$, in order to diſcover whether the reſult will be greater, or leſs, than 1881, or the abſolute term of the propoſed equation $x^3 - 39xx + 479x = 1881$, and conſequently whether 8.2 is greater, or leſs, than the true value of the leaſt root of that equation.

By this ſubſtitution we ſhall have $x^3 - 39xx + 479x$ $(= \overline{8.2}|^3 - 39 \times \overline{8.2}|^2 + 479 \times 8.2 = 551.368 - 39 \times 67.24 + 479 \times 8.2 = 551.368 - 2622.36 + 3927.8 = 4479.168 - 2622.36) = 1856.808$; which is leſs than 1881. Therefore 8.2 is leſs than the true value of the leaſt root of the equation $x^3 - 39xx + 479x = 1881$.

Art. 21.

$$x = 8.8 + \varphi,$$

and xx $(= \overline{8.8 + \varphi}|^2 = \overline{8.8}|^2 + 2 \times 8.8 \times \varphi + \&c) = 77.44 + 17.6 \times \varphi + \&c,$

and x^3 $(= \overline{8.8 + \varphi}|^3 = \overline{8.8}|^3 \times 3 \times \overline{8.8}|^2 \times \varphi + \&c = 681.472 + 3 \times 77.44 \times \varphi + \&c) = 681.472 + 232.32 \times \varphi + \&c,$

and $39xx$ $(= 39 \times \overline{77.44 + 17.6 \times \varphi + \&c} = 39 \times 77.44 + 39 \times 17.6 \times \varphi + \&c) = 3020.16 + 686.4 \times \varphi + \&c,$

and $479x$ $(= 479 \times \overline{8.8 + \varphi} = 479 \times 8.8 + 479 \times \varphi) = 4215.2 + 479 \times \varphi,$

and confequently $x^3 - 39xx + 479x =$

$$\left\{ \begin{array}{rlll} 681.472 & + 232.32 & \times \varphi & + \&c \\ - 3020.16 & - 686.4 & \times \varphi & - \&c \\ + 4215.2 & + 479 & \times \varphi & + \&c \end{array} \right\} =$$

$$\left\{ \begin{array}{rlll} 4896.672 & + 711.32 & \times \varphi & + \&c \\ - 3020.16 & - 686.4 & \times \varphi & - \&c \end{array} \right\}$$

$$= 1876.512 + 24.92 \times \varphi \quad \&c.$$

But $x^3 - 39xx + 479x$ is $= 1881$.

Therefore $1876.512 + 24.92 \times \varphi$ will alfo be $=$ 1881.

Therefore $24.92 \times \varphi$ will be $(= 1881 - 1876.512) = 4.488$; and confequently φ will be $= \frac{4.488}{24.92} = 0.18$.

Therefore

Therefore x, or $8.8 + \varphi$, will be $(= 8.8 + 0.18) = 8.98$, or the leaſt root of the propoſed equation $x^3 - 39xx + 479x = 1881$ will be $= 8.98$. Q. E. I.

Art. 24. This value of the ſaid leaſt root will be very little leſs than the truth. For, if we ſuppoſe x to be $= 8.98$, we ſhall have xx $(= \overline{8.98}|^2) = 80.6404$, and x^3 $(= \overline{8.98}|^3) = 724.150,792$, and $39xx$ $(= 39 \times 80.6404) = 3144.9796$, and $479x$ $(= 479 \times 8.98) = 4301.42$, and conſequently $x^3 - 39xx + 479x$ $(= 724.150,792 - 3144.9796 + 4301.42 = 5025.570,792 - 3144.979,6) = 1880.594,192$; which is very little leſs than 1881, or the abſolute term of the propoſed equation $x^3 - 39xx + 479x = 1881$. And conſequently 8.98 muſt be very little leſs than the true value of the leaſt root of the ſaid equation.

Art. 25. Since 8.98 is very little leſs than the true value of a, or the leaſt root of the propoſed equation $x^3 - 39xx + 479x = 1881$, it ſeems reaſonable to conjecture that it's true value may be the whole number 9. And ſo, upon trial, we ſhall find it to be. For, if we ſuppoſe x to be $= 9$, we ſhall have $xx = 81$, and $x^3 = 729$, and $39xx$ $(= 39 \times 81) = 3159$, and $479x$ $(= 479 \times 9) = 4311$, and conſequently $x^3 - 39xx + 479x$ $(= 729 - 3159 + 4311 = 5040 - 3159) = 1881$. Therefore 9 is the true value of a, or of the leaſt root of the propoſed equation $x^3 - 39xx + 479x = 1881$. Q. E. D.

Art. 26.

Art. 26. Having thus found one of the roots of the cubick equation $x^3 - 39xx + 479x = 1881$, we may reduce the equation to a lower degree, or convert it into a quadratick equation of which the roots ſhall be the ſame with the remaining roots of the ſaid cubick equation, by proceeding in the manner following:

Let a be put, as before, for 9, or the leaſt root of the ſaid equation; and let x repreſent each of the other two roots of it, if it has two other roots, or the only remaining root of it, if it has but one other root.

Then, ſince $a^3 - 39aa + 479a$ is $= 1881$, and $x^3 - 39xx + 479x$ is alſo $= 1881$, we ſhall have $a^3 - 39aa + 479a = x^3 - 39xx + 479x$, and conſequently (adding $39xx$ to both ſides,) $a^3 + 479a + 39xx - 39aa = x^3 + 479x$, and (ſubtracting $a^3 + 479a$ from both ſides of the equation,) $39xx - 39aa = x^3 - a^3 + 479x - 479a$, or $39 \times \overline{xx - aa} = x^3 - a^3 + 479 \times \overline{x - a}$, and (dividing both ſides of the equation by the reſidual quantity $x - a$,) $39 \times \overline{x + a} = xx + xa + aa + 479$, or $39x + 39a = xx + ax + aa + 479$, and (ſubtracting $xx + ax$ from both ſides,) $39x + 39a - ax - xx = aa + 479$, and (ſubtracting $39a$ from both ſides,) $39x - ax - xx = 479 + aa - 39a$, or (becauſe a is $= 9$,) $39x - 9x - xx = 479 + 81 - 39 \times 9$, or $30x - xx$ ($= 560 - 39 \times 9 = 560 - 351$) $= 209$; which is only a quadratick equation.

Q. E. F.

Art. 27

Art. 27. This quadratick equation, $30x - xx = 209$, will have two roots, which may be found in the manner following:

Subtract both sides of the equation $30x - xx = 209$ from 225, or the square of 15, or of $\frac{30}{2}$, or of half the co-efficient of x in the said equation. And we shall have $225 - 30x + xx$ ($= 225 - 209$) $= 16$. Therefore the square-root of the trinomial quantity $225 - 30x + xx$ will be equal to the square-root of 16, that is, to 4. But the trinomial quantity $225 - 30x + xx$ has two square-roots, to wit, $15 - x$ and $x - 15$. Therefore $15 - x$ will be $= 4$, and consequently 15 will be $= 4 + x$, and x will be $= 15 - 4 = 11$. And $x - 15$ will also be $= 4$, and consequently x will be $= 4 + 15 = 19$. Therefore 11 and 19 will be the two roots of the quadratick equation $30x - xx = 209$, and consequently will be the middle and greatest roots of the proposed cubick equation $x^3 - 39xx + 479x = 1881$.

Q. E. I.

And so, upon trial, we shall find them to be. For, if we suppose x to be $= 11$, we shall have $xx = 121$, and $x^3 = 1331$, and $39xx$ ($= 39 \times 121$) $= 4719$, and $479x$ ($= 479 \times 11$) $= 5269$, and consequently $x^3 - 39xx + 479x$ ($= 1331 - 4719 + 5269 = 6600 - 4719$) $= 1881$.

And, if we suppose x to be $= 19$, we shall have $xx = 361$, and $x^3 = 6859$, and $39xx$ ($= 39 \times 361$) $=$

$= 14,079$, and $479x$ $(= 479 \times 19) = 9101$, and conſequently $x^3 - 39xx + 479x$ $(= 6859 - 14,079 + 9101 = 15,960 - 14,079) = 1881$.

Therefore both 11 and 19 are roots of the propoſed equation $x^3 - 39xx + 479x = 1881$, and conſequently the three roots of that equation are 9, 11, and 19.

Q. E. I.

Another Example of the Reſolution of an Equation that, by the Form of it, or the Changes of the Signs + and — prefixed to it's Terms, ſeems capable of having more than One real and affirmative Root, by means of the foregoing Proceſſes deſcribed above in Art. 9, 10, 11, 12, *and* 13.

Art. 28. Let it be required to reſolve the equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$ (which riſes to the fifth power of x,) by means of the ſaid proceſſes.

This equation is of ſuch a form as to be capable of having three real and affirmative roots, if the co-efficients of x^5, x^4, x^3, and x, to wit, 1, 7, 20, and 155, and the abſolute term 10,000, are of the proper relative magnitudes with reſpect to each other for that purpoſe. But of theſe roots, (whether they be three in number, or fewer than three,) I now propoſe to inveſtigate only one root;

root; and that will be the leaſt of them, if there are more than one: and this root I ſhall denote by the letter a.

Art. 29. Now, ſince $x^5 - 7x^4 + 20x^3 - 155xx$ is $= 10,000$, we ſhall have $x^5 + 20x^3 = 10,000 + 7x^4 + 155xx$; and conſequently (becauſe a is equal to one of the values of x, to wit, the leaſt value of it, if it has more than one value,) $a^5 + 20a^3$ will be $= 10,000 + 7a^4 + 155aa$. We muſt therefore endeavour to find the root a of this equation $a^5 + 20a^3 = 10,000 + 7a^4 + 155aa$.

Now the root of this equation will be greater than the root of the equation $y^5 + 20y^3 = 10,000$, becauſe the trinomial quantity $10,000 + 7a^4 + 155aa$ is greater than the ſingle quantity 10,000. We will therefore, as a firſt approximation to the value of a, or the root of the equation $a^5 + 20a^3 = 10,000 + 7a^4 + 155aa$, ſeek the value of y, or the root of the equation $y^5 + 20y^3 = 10,000$.

Art. 30. Now, if we ſuppoſe y to be equal to 1, we ſhall have $y^3 = 1$, and $y^5 = 1$, and $20y^3$ $(= 20 \times 1)$ $= 20$, and conſequently $y^5 + 20y^3$ $(= 1 + 20) = 21$; which is much leſs than the abſolute term 10,000. Therefore 1 muſt be much leſs than the value of y in the equation $y^5 + 20y^3 = 10,000$.

Secondly, if we ſuppoſe y to be $= 10$, we ſhall have $y^3 = 1000$, and $y^5 = 100,000$, and $20y^3 = 20,000$, and

$y^5 +$

$y^5 + 20y^3$ ($= 100,000 + 20,000$) $= 120,000$; which is much greater than the abſolute term 10,000. Therefore 10 muſt be much greater than the value of y in the equation $y^5 + 20y^3 = 10,000$.

We will therefore, in the 3d place, ſuppoſe y to be equal to 5. And then we ſhall have $y^3 = 125$, and $y^5 = 3125$, and $20y^3$ ($= 20 \times 125$) $= 2500$, and $y^5 + 20y^3$ ($= 3125 + 2500$) $= 5625$; which is leſs than the abſolute term 10,000. Therefore 5 muſt be leſs than the value of y in the equation $y^5 + 20y^3 = 10,000$.

We will therefore, in the 4th place, ſuppoſe y to be equal to 6. And then we ſhall have $y^3 = 216$, and $y^5 = 7776$, and $20y^3$ ($= 20 \times 216$) $= 4320$, and conſequently $y^5 + 20y^3$ ($= 7776 + 4320$) $= 12,096$; which is greater than the abſolute term 10,000. Therefore 6 muſt be greater than the value of y in the equation $y^5 + 20y^3 = 10,000$.

We will therefore, in the fifth place, ſuppoſe y to be equal to 5.8. And then we ſhall have $y^3 = 195.112$, and $y^5 = 6563.56768$, and $20y^3$ ($= 20 \times 195.112$) $= 3902.240$, and conſequently $y^5 + 20y^3$ ($= 6563.56763 + 3902.240$) $= 10,465.80768$; which is ſomewhat greater than the abſolute term 10,000. Therefore 5.8 muſt be ſomewhat greater than the value of y in the equation $y^5 + 20y^3 = 10,000$. But the difference will be but ſmall, and conſequently we may conſider 5.8 as the root of the ſaid equation $y^5 + 20y^3 = 10,000$, and

as

as the first near value of a in the equation $a^5 + 20a^3 = 10,000 + 7a^4 + 155aa$.

Art. 31. Let this quantity 5.8 be called α. And let us, for a second near value of a, find the root of the equation $y^5 + 20y^3 = 10,000 + 7\alpha^4 + 155\alpha^2$. For this root will be greater than α, or 5.8, or the root of the equation $y^5 + 20y^3 = 10,000$, because the trinomial quantity $10,000 + 7\alpha^4 + 155\alpha^2$ is greater than the single quantity 10,000; but it will be less than a, or the root of the equation $a^5 + 20a^3 = 10,000 + 7a^4 + 155aa$, because the trinomial quantity $10,000 + 7\alpha^4 + 155\alpha^2$ is less than the trinomial quantity $10,000 + 7a^4 + 155aa$.

Now, since α is $= 5.8$, we shall have α^2 $(= \overline{5.8}|^2) = 33.64$, and α^4 $(= \overline{5.8}|^4) = 1131.6496$, and $7\alpha^4$ $(= 7 \times 1131.6496) = 7921.5472$, and $155\alpha^2$ $(= 155 \times 33.64) = 1850.20$, and consequently $10,000 + 7\alpha^4 + 155\alpha^2$ $(= 10,000 + 7921.5472 + 1850.20) = 19,1771.7472$. Therefore we shall have $y^5 + 20y^3$ $(= 10,000 + 7\alpha^4 + 155\alpha^2) = 19,771.7472$; of which equation we must now endeavour to find the root.

Art. 32. Now we have seen above, that, if y be supposed to be $= 6$, the compound quantity $y^5 + 20y^3$ will be $= 12,096$; which is less than the absolute term, 19,771.7472, of the present equation $y^5 + 20y^3 = 19,771.7472$. Therefore 6 must be less than the root of the said equation.

We

We will therefore ſuppoſe y to be equal to 7. And then we ſhall have $y^3 = 343$, and $y^5 = 16,807$, and conſequently $20y^3$ ($= 20 \times 343$) $= 6860$, and $y^5 + 20y^3$ ($= 16,807 + 6860$) $= 23,667$; which is greater than the abſolute term 19,771.7472. Therefore 7 muſt be greater than the root of the equation $y^5 + 20y^3 = 19,771.7472$.

We will therefore, in the next place, ſuppoſe y to be equal to 6.7. And then we ſhall have $y^3 = 300.763$, and $y^5 = 13,501.25107$, and conſequently $20y^3$ ($= 20 \times 300.763$) $= 6015.260$, and $y^5 + 20y^3$ ($= 13,501.25107 + 6015.260$) $= 19,516.51107$; which is very little leſs than the abſolute term 19,771.7472. Therefore 6.7 muſt be very little leſs than the root of the equation $y^5 + 20y^3 = 19,771.7472$. We may therefore conſider 6.7 as the ſecond near value of a, or of the root of the equation $a^5 + 20a^3 = 10,000 + 7a^4 + 155aa$, or of the equation $a^5 - 7a^4 + 20a^3 - 155aa = 10,000$, or of the leaſt root of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$.

Art. 33. This ſecond near value of a, or of the leaſt root of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$, is conſiderably leſs than it's true value. For, if we ſuppoſe x to be equal only to this ſecond near value of a, to wit, 6.7, the two terms $x^5 + 20x^3$ will be leſs, inſtead of being greater, than the two terms $7x^4 + 155xx$, which are to be ſubtracted from them. For, upon this ſuppoſition we ſhall have $xx = 44.89$, and $x^3 = 300.763$, and $x^4 = 2015.1121$, and $x^5 = 13,501.25107$,

.25107, and $7x^4$ ($= 7 \times 2015.1121$) $= 14,105.7847$, and $20x^3$ ($= 20 \times 300.763$) $= 6015.260$, and $155xx$ ($= 155 \times 44.89$) $= 6957.95$, and $x^5 + 20x^3$ ($= 13,501.25107 + 6015.260$) $= 19,516.51107$, and $7x^4 + 155xx$ ($= 14,105.7847 + 6957.95$) $= 21,063.7347$; which is greater than $19,516.51107$, or $x^5 + 20x^3$.

Therefore x must be considerably greater than 6.7, in order to make $x^5 + 20x^3$ become, not only greater than $7x^4 + 155xx$, but greater by an excess equal to 10,000.

We will therefore suppose x to be equal to 8. And then we shall have $xx = 64$, and $x^3 = 512$, and $x^4 = 4096$, and $x^5 = 32768$, and consequently $7x^4$ ($= 7 \times 4096$) $= 28672$, and $20x^3$ ($= 20 \times 512$) $= 10240$, and $155xx$ ($= 155 \times 64$) $= 9920$, and $x^5 - 7x^4 + 20x^3 - 155xx$ ($= 32768 - 28672 + 10240 - 9920 = 43,008 - 38,592$) $= 4416$; which is less than 10,000, or the absolute term of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$. Therefore 8 will be somewhat less than the least root of the said equation.

We will therefore now suppose x to be $= 8.5$. And then we shall have $xx = 72.25$, and $x^3 = 614.125$, and $x^4 = 5220.0625$, and $x^5 = 44370.53125$, and consequently $7x^4$ ($= 7 \times 5220.0625$) $= 36540.4375$, and $20x^3$ ($= 20 \times 614.125$) $= 12282.500$, and $155xx$ ($= 155 \times 72.25$) $= 11,198.75$, and $x^5 - 7x^4 + 20x^3 - 155xx$ ($= 44,370.53125 - 36,540.4375 + 12,282.500 - 11,198.75 = 56,653.03125 - 47,739.1875$)

.1875) = 8913.84375; which is lefs than the abfolute term 10,000. Therefore 8.5 will be lefs than the leaft root of the propofed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$. But the difference between them will not be great, becaufe 8913.84375 is not a great deal lefs than 10,000. Therefore 8.5 will be fufficiently near to the true value of the faid leaft root to be the ground of a further approximation to it by Mr. Raphfon's method.

Art. 34. Let us therefore fuppofe x to be $= 8.5 + z$.

Then we fhall have

$$xx\ (= \overline{8.5+z}|^2 = \overline{8.5}|^2 + 2 \times 8.5 \times z + \&c)$$
$$= 72.25 + 17.0 \times z + \&c,$$

$$\text{and } x^3\ (= \overline{8.5+z}|^3 = \overline{8.5}|^3 + 3 \times \overline{8.5}|^2 \times z + \&c$$
$$= 614.125 + 3 \times 72.25 \times z + \&c)$$
$$= 614.125 + 216.75 \times z + \&c,$$

$$\text{and } x^4\ (= \overline{8.5+z}|^4 = \overline{8.5}|^4 + 4 \times \overline{8.5}|^3 \times z + \&c$$
$$= 5220.0625 + 4 \times 614.125 \times z + \&c)$$
$$= 5220.0625 + 2456.500 \times z + \&c,$$

$$\text{and } x^5\ (= \overline{8.5+z}|^5 = \overline{8.5}|^5 + 5 \times \overline{8.5}|^4 \times z + \&c$$
$$= 44{,}370.53125 + 5 \times 5220.0625 \times z$$
$$+ \&c) = 44{,}370.53125 + 26{,}100.3125$$
$$\times z + \&c.$$

Therefore

Therefore $7x^4$ will be ($= 7 \times \overline{5220.0625 + 2456.500 \times z + \&c} = 7 \times 5220.0625 + 7 \times 2456.500 \times z + \&c) = 36540.4375 + 17195.500 \times z + \&c$, and $20x^3$ will be ($= 20 \times \overline{614.125 + 216.75 \times z + \&c} = 20 \times 614.125 + 20 \times 216.75 \times z + \&c) = 12{,}282.500 + 4335.00 \times z + \&c$, and $155xx$ will be ($= 155 \times \overline{72.25 + 17.0 \times z + \&c} = 155 \times 72.25 + 155 \times 17.0 \times z + \&c) = 11{,}198.75 + 2635.0 \times z + \&c$, and consequently $x^5 - 7x^4 + 20x^3 - 155xx$ will be $=$

$$\left\{ \begin{array}{rrrl} & 44{,}370.53125 & + \; 26{,}100.3125 & \times z + \&c \\ - & 36{,}540.4375 & - \; 17{,}195.500 & \times z - \&c \\ + & 12{,}282.500 & + \; 4335.00 & \times z + \&c \\ - & 11{,}198.75 & - \; 2635.0 & \times z - \&c \end{array} \right\}$$

$$= \left\{ \begin{array}{rrrl} & 56{,}653.03125 & + \; 30{,}435.3125 & \times z + \&c \\ - & 47{,}739.1875 & - \; 19{,}830.500 & \times z - \&c \end{array} \right\}$$

$$= 8{,}913.84375 + 10{,}604.8125 \times z \;\; \&c.$$

But $x^5 - 7x^4 + 20x^3 - 155xx$ is $= 10{,}000$.

Therefore $8{,}913.84375 + 10{,}604.8125 \times z + \&c$ will also be $= 10{,}000$.

And consequently $10{,}604.8125 \times z$ will be ($= 10{,}000 - 8{,}913.84375$) $= 1086.15625$, and z will be $= \dfrac{1086.15625}{10{,}604.8125} = 0.10$.

Therefore

Therefore $8.5 + z$ will be $(= 8.5 + 0.10) = 8.60$, or 8.6, and consequently x, or the least root of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$ will be, nearly, $= 8.60$, or 8.6. Q. E. I.

Art. 35. Now let 8.6 be substituted instead of x in the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$. And we shall have xx $(= \overline{8.6}|^2) = 73.96$, and $x^3 = 636.056$, and $x^4 = 5470.0816$, and $x^5 = 47{,}042.701{,}76$, and consequently $7x^4$ $(= 7 \times 5470.0816) = 38{,}290.5712$, and $20x^3$ $(= 20 \times 636.056) = 12{,}721.120$, and $155xx$ $(= 155 \times 73.96) = 11{,}463.80$, and $x^5 - 7x^4 + 20x^3 - 155xx$ $(= 47{,}042.701{,}76 - 38{,}290.5712 + 12{,}721.120 - 11{,}463.80 = 59{,}763.821{,}76 - 49{,}754.3712) = 10{,}009.450{,}56$; which is a little greater than the absolute term 10,000. Therefore 8.6 must be a little greater than the true value of the least root of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$.

Art. 36. We have now seen that, when x is $= 8.5$, the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$ will be equal to 8913.84375; and that, when x is $= 8.6$, the said compound quantity will be equal to 10,009.45056. Therefore, while x increases from 8.5 to 8.6, the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$ will increase from 8913.84375 to 10,009.45056. Therefore there will be some point of time, during the increase of x from 8.5 to 8.6, at which the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$ will be equal to 10,000, which is of an intermediate

intermediate magnitude between 8913.84375 and 10,009.45056; or, in other words, the root of the equation $x^5 - 7x^4 + 20x^3 - 155xx$ will be of an intermediate magnitude between 8.5 and 8.6. But it will be much nearer to 8.6 than to 8.5, becaufe 10,009.45056 is much nearer than 8913.84375 to 10,000.

We will therefore fuppofe x to be $= 8.6 - v$.

And then we fhall have

xx $(= \overline{8.6-v}|^2 = \overline{8.6}|^2 - 2 \times 8.6 \times v + \&c) = 73.96 - 17.2 \times v + \&c,$

and x^3 $(= \overline{8.6-v}|^3 = \overline{8.6}|^3 - 3 \times \overline{8.6}|^2 \times v + \&c = 636.056 - 3 \times 73.96 \times v + \&c) = 636.056 - 221.88 \times v + \&c,$

and x^4 $(= \overline{8.6-v}|^4 = \overline{8.6}|^4 - 4 \times \overline{8.6}|^3 \times v + \&c = 5470.0816 - 4 \times 636.056 \times v + \&c) = 5470.0816 - 2544.224 \times v + \&c,$

and x^5 $(= \overline{8.6-v}|^5 = \overline{8.6}|^5 - 5 \times \overline{8.6}|^4 \times v + \&c = 47,042.70176 - 5 \times 5470.0816 \times v + \&c) = 47,042.70176 - 28350.4080 \times v + \&c,$

and confequently $7x^4$ $(= 7 \times \overline{5470.0816 - 2544.224 \times v + \&c} = 7 \times 5470.0816 - 7 \times 2544.224 \times v + \&c) = 38,290.5712 - 17,809.568 \times v + \&c,$

and

and $20x^3$ (= $20 \times \overline{636.056 - 221.88 \times v + \&c} =$

$20 \times 636.056 - 20 \times 221.88 \times v + \&c)$

$12,721.120 - 4437.60 \times v + \&c,$

and $155xx$ (= $155 \times \overline{73.96 - 17.2 \times v \times \&c} =$

$155 \times 73.96 - 155 \times 17.2 \times v + \&c)$

$= 11,463.80 - 2666.0 \times v + \&c,$

and $x^5 - 7x^4 + 20x^3 - 155xx =$

$$\left\{\begin{array}{rrl} 47,042.70176 & - \ 28,350.4080 & \times v + \&c \\ - \ 38,296.5712 & + \ 17,809.568 & \times v - \&c \\ + \ 12,721.120 & - \ 4,437.60 & \times v + \&c \\ - \ 11,463.80 & + \ 2,666.0 & \times v - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{rrl} 59,763.821,76 & - \ 32,788.0080 & \times v + \&c \\ - \ 49,754.371,2 & + \ 20,475.568 & \times v - \&c \end{array}\right\}$$

$$= 10,009.450,56 - 12,312.4400 \times v \ \&c.$$

But $x^5 - 7x^4 + 20x^3 - 155xx$ is $= 10,000$.

Therefore $10,009.450,56 - 12,312.4400 \times v$ will likewiſe be $= 10,000$.

And conſequently $10,009.450,56$ will be $= 10,000 + 12,312.4400 \times v$, and $12,312.4400 \times v$ will be $(= 10,009.450,56 - 10,000) = 9.450,56$, and v will be $=$

$$\frac{9.450,56}{12,312.4400} = 0.000,767,5.$$

Therefore

Therefore 8.6 — v will be (= 8.6 — 0.000,767,5) = 8.599,232,5; and consequently x, or the least root of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,060$, will be = 8.599,232,5. Q. E. I.

Art. 37. This number 8.599,232,5 is exact in either all it's figures, or all but the last figure 5. But the trial of it by substituting it instead of x in the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$ would be attended with a good deal of labour. However, it may be performed with a tolerable degree of exactness, though not with perfect exactness, by the help of logarithms, without any great trouble, in the manner following:

The logarithm of 8.599,200,0 is 0.934,458,0, and the logarithm of 8,599,300,0 is 0.934,463,1, which exceeds the former logarithm by 0.000,005,1. Therefore the logarithm of 8.599,232,5 will be of an intermediate magnitude between 0.934,458,0 and 0.934,463,1, and its excess above 0.934,458,0 is to be found by the following proportion, to wit, As 8.599,300,0 — 8.599,200,0 is to 8.599,232,5 — 8.599,200,0, so is 0.934,463,1 — 0.934,458,0 to the excess of the logarithm of 8.599,232,5 above 0.934,458,0, that is, as 0.000,100,0 is to 0.000,032,5, so is 0.000,005,1 to the said excess, or as 1000 to 325, so is 0.000,005,1 to the said excess. Therefore the said excess will be $= \frac{325 \times 0.000,005,1}{1000} = \frac{0.001,657,5}{1000}$ = 0.000,001,6; and consequently the logarithm of 8.599,232,5 will be (= 0.934,458,0 + 0.000,001,6) =

$= 0.934,459,6$. Therefore, if x be supposed to be equal to $8.599,232,5$, the logarithm of xx will be $(= 2 \times 0.934,459,6) = 1.868,919,2$, and the logarithm of x^3 will be $(= 3 \times 0.934,459,6) = 2.803,378,8$, and the logarithm of x^4 will be $(= 4 \times 0.934,459,6) = 3.737,838,4$, and the logarithm of x^5 will be $(= 5 \times 0.934,459,6) = 4.672,298,0$. Therefore xx will be $= 73.946,762,7$, and x^3 will be $= 635.885,294,1$; and x^4 will be $= 5468.125,000,0$, and x^5 will be $= 47,021.663,043,4$. Therefore $7x^4$ will be $(= 7 \times 5468.125,000,0) = 38,276.875,000,0$, and $20x^3$ will be $(= 20 \times 635.885,294,1) = 12,717.705,882,0$, and $155xx$ will be $(= 155 \times 73.946,762,7) = 11,461.748,218,5$, and $x^5 - 7x^4 + 20x^3 - 155xx$ will be $(= 47,021.663,043,4 - 38,276.875,000,0 + 12,717.705,882,0 - 11,461.748,218,5 = 59,739.368,925,4 - 49,738.623,218,5) = 10,000.745,706,9$; which exceeds 10,000, or the absolute term of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$, by less than an unit, or the 10,000th part of the said absolute term. Therefore $8.599,232,5$ must be very nearly equal to the root of the said equation. Q. E. D.

Art. 38. This number $8.599,232,5$ is the only root of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$. For, if we should suppose it to have any other root greater than a, or $8.599,232,5$, an impossible conclusion would follow from such a supposition; as may be shewn in the manner following:

Let

Let b denote the other and greater root of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$, if it can have such other root.

Then will the compound quantity $b^5 - 7b^4 + 20b^3 - 155bb$ be $= 10,000$; and consequently it will also be equal to $a^5 - 7a^4 + 20a^3 - 155aa$. Therefore (adding $7b^4$ and $155bb$ to both sides,) we shall have $b^5 + 20b^3 = a^5 + 7b^4 - 7a^4 + 20a^3 + 155bb - 155aa$; and (subtracting $a^5 + 20a^3$ from both sides) we shall have $b^5 - a^5 + 20b^3 - 20a^3 = 7b^4 - 7a^4 + 155bb - 155aa$, or $b^5 - a^5 + 20 \times \overline{b^3 - a^3} = 7 \times \overline{b^4 - a^4} + 155 \times \overline{bb - aa}$, and consequently (dividing both sides of the equation by $b - a$,) $b^4 + b^3a + b^2a^2 + ba^3 + a^4 + 20 \times \overline{bb + ba + aa} = 7 \times \overline{b^3 + b^2a + ba^2 + a^3} + 155 \times \overline{b + a}$, or $b^4 + b^3a + b^2a^2 + ba^3 + a^4 + 20bb + 20ba + 20aa = 7b^3 + 7b^2a + 7ba^2 + 7a^3 + 155b + 155a$. Therefore, if we subtract $7a^3 + 155a$ from both sides of this equation, we shall have $7b^3 + 7b^2a + 7ba^2 + 155b = b^4 + b^3a + b^2a^2 + ba^3 + 20bb + 20ba + a^4 - 7a^3 + 20aa - 155a$; and, if we subtract $b^4 + b^3a + b^2a^2 + ba^3 + 20bb + 20ba$ from both sides, we shall have $7b^3 + 7b^2a + 7ba^2 + 155b - b^4 - b^3a - b^2a^2 - ba^3 - 20bb - 20ba = a^4 - 7a^3 + 20aa - 155a$, or

$$\left.\begin{array}{l} 155b - 20bb + 7b^3 - b^4 \\ - 20ab + 7abb - ab^3 \\ + 7a^2b - a^2bb \\ - a^3b \end{array}\right\}$$

$= a^4 - 7a^3 + 20aa - 155a = \frac{a^5 - 7a^4 + 20aa + 155aa}{a}$

$= \frac{10,000}{a}$, or (if we fubftitute 8.6 for a, to which it is very nearly equal,)

$$\left\{\begin{array}{llll} & 155b & - \ 20bb & + \ 7b^3 - b^4 \\ - & 20 \times 8.6b & + \ 7 \times 8.6bb & - \ 8.6b^3 \\ + & 7 \times 73.96 \times b & - \ 73.96bb & \\ - & 636.056 \times b & & \end{array}\right\}$$

$= \frac{10,000}{8.6} = 1162.7$, or

$$\left\{\begin{array}{llll} & 155b & - \ 20bb & + \ 7 \times b^3 - b^4 \\ - & 172.0 \times b & + \ 60.2bb & - \ 8.6 \times b^3 \\ + & 517.72 \times b & - \ 73.96bb & \\ - & 636.056 \times b & & \end{array}\right\}$$

$= 1162.7$, or

$$\left\{\begin{array}{llll} & 672.72 \times b & - \ 93.96bb + & 7b^3 \quad - b^4 \\ - & 808.056 \times b & + \ 60.2\ bb & - \ 8.6 \times b^3 \end{array}\right\}$$

$= 1162.7$; that is, if from the three terms $672.72 \times b + 60.2 \times bb + 7b^3$ we fubtract the four terms $808.056 \times b + 93.96 \times bb + 8.6 \times b^3 + b^4$, the remainder will be equal to 1162.7. But the faid four terms $808.056 \times b + 93.96 \times bb + 8.6 \times b^3 + b^4$, are greater than the faid three terms $672.72 \times b + 60.2 \times bb + 7b^3$, and therefore cannot be fubtracted from them. Therefore the fuppofition, from which it followed that

that the said four terms would be less than the said three terms, and might be subtracted from them, and that the remainder would be equal to 1162.7, or the supposition that the equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$ had a second real and affirmative root, b, that was greater than a, or 8.6, could not be a true supposition. We may therefore conclude that 8.6, or 8.599,232,5, is the only root of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$. Q. E. D.

Another Example of the Resolution of an Equation that, by the Form of it, or the Changes of the Signs + and — prefixed to it's Terms, seems capable of having more than One real and affirmative Root, by means of the foregoing Processes described above in Art. 9, 10, 11, 12, and 13.

Art. 39. Let it be required to resolve the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x = 5$, (which rises to the eighth power of x,) by means of the said processes.

This equation is taken from Sir Isaac Newton's *Arithmetica Universalis*, page 276 of the second edition, and is the highest numeral equation mentioned in that learned treatise.

treatise. It is of such a form as to be capable of having three real and affirmative roots, if the co-efficients of the several powers of x, to wit, 1, 4, 1, 10, 5,5, 10, and 10, and the absolute term 5, are of the proper relative magnitudes, with respect to each other, for that purpose. But of these roots (whether they be three in number, or fewer than three,) I now propose to investigate only one root; and that will be the least of them, if there are more than one: and this root I shall denote by the letter a.

Art. 40. Now, since $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x$ is $= 5$, we shall have $x^8 + 4x^7 + 5x^4 = 5 + x^6 + 10x^5 + 5x^3 + 10xx + 10x$. But a is equal to one of the values of x, to wit, the least. Therefore we shall also have $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10aa + 10a$. We must therefore endeavour to find the root a of this equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10aa + 10a$.

Now the root of this equation will be greater than the root of the equation $y^8 + 4y^7 + 5y^4 = 5$, because the sextinomial quantity $5 + a^6 + 10a^5 + 5a^3 + 10aa + 10a$ is greater than the single quantity 5. We will therefore, as a first approximation to the value of a, or the root of the equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, seek the value of y, or the root of the equation $y^8 + 4y^7 + 5y^4 = 5$.

Art. 41. Now, if we suppose y to be $= 1$, we shall have $y^4 = 1$, and $y^7 = 1$, and $y^8 = 1$, and consequently

y^8

$y^8 + 4y^7 + 5y^4$ $(= 1 + 4 + 5) = 10$; which is greater than the abſolute term 5. Therefore 1 muſt be greater than the true value of y in the equation $y^8 + 4y^7 + 5y^4 = 5$.

We will therefore, in the ſecond place, ſuppoſe y to be $= 0.7$.

And then we ſhall have $yy = 0.49$, and $y^3 = 0.343$, and $y^4 = 0.2401$, and $y^5 = 0.168,07$, and $y^6 = 0.117,649$, and $y^7 = 0.082,354,3$, and $y^8 = 0.057,648,01$, and conſequently $4y^7$ $(= 4 \times 0.082,354,3) = 0.329,417,2$, and $5y^4$ $(= 5 \times 0.2401) = 1.2005$, and $y^8 + 4y^7 + 5y^4$ $(= 0.057,648,01 + 0.329,417,2 + 1.2005)$ $= 1.587,565,31$; which is leſs than the abſolute term 5. Therefore 0.7 muſt be leſs than the value of y in the equation $y^8 + 4y^7 + 5y^4 = 5$.

We will therefore, in the 3d place, ſuppoſe y to be $= 0.9$.

And then we ſhall have $yy = 0.81$, and $y^3 = 0.729$, and $y^4 = 0.6561$, and $y^5 = 0.59049$, and $y^6 = 0.531,441$, and $y^7 = 0.478,296,9$, and $y^8 = 0.430,467,21$, and conſequently $4y^7$ $(= 4 \times 0.478,296,9) = 1.913,187,6$, and $5y^4$ $(= 5 \times 0.6561) = 3.2800$, and $y^8 + 4y^7 + 5y^4$ $(= 0.430,467,21 + 1.913,187,6 + 3.2800)$ $= 5.623,654,81$; which is ſomewhat greater than the abſolute term 5. Therefore 0.9 muſt be ſomewhat greater than the root of the equation $y^8 + 4y^7 + 5y^4$ $= 5$.

$= 5$. But the difference will not be great; and consequently we may consider this quantity 0.9 as the first near value of a, or of the root of the equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, of which we are in search.

Art. 42. Now let α be put for 0.9, or the said first near value of a. And, as a second near value of it, let us seek the root of the equation $y^8 + 4y^7 + 5y^4 = 5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$; which will be greater than α, but less than a, because the sextinomial quantity $5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$ is greater than the single quantity 5, but less than the sextinomial quantity $5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$.

Now, because α is $= 0.9$, we shall have α^6 $(= \overline{0.9}|^6)$ $= 0.531,441$, and $10\alpha^5$ $(= 10 \times 0.59049) = 5.9049$, and $5\alpha^3$ $(= 5 \times 0.729) = 3.645$, and $10\alpha^2$ $(= 10 \times 0.81)$ $= 8.1$, and 10α $(= 10 \times 0.9) = 9$, and consequently $5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$ $(= 5 + 0.531,441 + 5.9049 + 3.645 + 8.1 + 9) = 32.181,341$; and therefore the equation $y^8 + 4y^7 + 5y^4 = 5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$ will be $y^8 + 4y^7 + 5y^4 = 32.181,341$, or (neglecting the decimal fraction 0.181, 341,) $y^8 + 4y^7 + 5y^4 = 32$. Of this equation we are now to find the root.

Art. 43. Now let y (which we know to be greater than α, or 0.9,) be supposed to be $= 1.2$.

Then

Then we shall have $yy = 1.44$, and $y^3 = 1.728$, and $y^4 = 2.0736$, and $y^5 = 2.488,32$, and $y^6 = 2.985,984$, and $y^7 = 3.583,180,8$, and $y^8 = 4.299,816,96$, and consequently $4y^7$ ($= 4 \times 3.583,180,8$) $= 14.332,723,2$, and $5y^4$ ($= 5 \times 2.0736$) $= 10.3680$, and $y^8 + 4y^7 + 5y^4$ ($= 4.299,816,96 + 14.332,723,2 + 10.3680$) $= 29.000,540,16$; which is somewhat less than the absolute term 32. Therefore 1.2 is somewhat less than the root of the equation $y^8 + 4y^7 + 5y^4 = 32$. But the difference between them is not great. And therefore we may consider 1.2 as the second near value of the root of the equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, of which we are in search.

Art. 44. Now let β be put for 1.2, or this second near value of the root a of the said equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$. And for a third near value of it, let us seek the root of the equation $y^8 + 4y^7 + 5y^4 = 5 + \beta^6 + 10\beta^5 + 5\beta^3 + 10\beta^2 + 10\beta$; which will be greater than β, or the root of the equation $y^8 + 4y^7 + 5y^4 = 32$, or $y^8 + 4y^7 + 5y^4 = 5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$, but less than a, or the root of the equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, because the sextinomial quantity $5 + \beta^6 + 10\beta^5 + 5\beta^3 + 10\beta^2 + 10\beta$ is greater than the sextinomial quantity $5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$, but less than the sextinomial quantity $5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$.

Now, because β is $= 1.2$, we shall have β^6 ($= 1.2|^6$) $= 2.985,984$, and $10\beta^5$ ($= 10 \times 1.2|^5 = 10 \times 2.488,32$)

$2.488,32) = 24.8832$, and $5\varsigma^3$ $(= 5 \times \overline{1.2}|^3 = 5 \times 1.728) = 8.640$, and $10\varsigma^2$ $(= 10 \times \overline{1.2}|^2 = 10 \times 1.44) = 14.4$, and 10ς $(= 10 \times 1.2) = 12$, and confequently $5 + \varsigma^6 + 10\varsigma^5 + 5\varsigma^3 + 10\varsigma^2 + 10\varsigma$ $(= 5 + 2.985,984 + 24.8832 + 8.640 + 14.4 + 12) = 67.909,184$. Therefore the equation $y^8 + 4y^7 + 5y^4 = 5 + \varsigma^6 + 10\varsigma^5 + 5\varsigma^3 + 10\varsigma^2 + 10\varsigma$ will be $y^8 + 4y^7 + 5y^4 = 67.909,184$, or, nearly, $y^8 + 4y^7 + 5y^4 = 68$. This equation we muft now endeavour to refolve.

Art. 45. Now let y (which we know to be greater than ς, or 1.2,) be fuppofed to be $= 1.3$. Then we fhall have yy $(= \overline{1.3}|^2) = 1.69$, and y^3 $(= \overline{1.3}|^3) = 2.197$, and $y^4 = 2.8561$, and $y^5 = 3.712,93$, and $y^6 = 4.826,809$, and $y^7 = 6.274,851,7$, and $y^8 = 8.157,306,82$, and confequently $4y^7$ $(= 4 \times 6.274,851,7) = 26.099,406,8$, and $5y^4$ $(= 5 \times 2.8561) = 14.2805$, and $y^8 + 4y^7 + 5y^4$ $(= 8.157,306,82 + 26.099,406,8 + 14.2805) = 48.537,213,62$; which is lefs than the abfolute term 68. Therefore 1.3 will be lefs than the root of the equation $y^8 + 4y^7 + 5y^4 = 68$.

Now let y be fuppofed to be $= 1.4$.

And then we fhall have $yy = 1.96$, and $y^3 = 2.744$, and $y^4 = 3.8416$, and $y^5 = 5.37824$, and $y^6 = 7.529,536$, and $y^7 = 10.541,350,4$, and $y^8 = 14.757,890,56$, and confequently $4y^7$ $(= 4 \times 10.541,350,4) = 42.165,401,6$, and $5y^4$ $(= 5 \times 3.8416) = 19.2080$, and $y^8 +$

$4y^7$

$4y^7 + 5y^4$ (= 14.757,890,56 + 42.165,401,6 + 19.2080) = 76.131,292,16; which is greater than the abſolute term 68. Therefore 1.4 will be greater than the root of the equation $y^8 + 7y^7 + 5y^4 = 68$. But the difference will not be great. And therefore we may conſider 1.4 as a third near value of the root of the equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, or as a third near value of the leaſt root (if it has more than one root,) of the equation $x^8 + 4x^7 + 5x^4 = 5 + x^6 + 10x^5 + 5x^3 + 10x^2 + 10x$, or of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$.

Art. 46. Now let 1.4 be ſubſtituted inſtead of x in the compound quantity $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$; and it will be = 14.757,890,56 + 4 × 10.541,350,4 — 7.529,536 — 10 × 5.37824 + 5 × 3.8416 — 5 × 2.744 — 10 × 1.96 — 10 × 1.4 = 14.757,890,56 + 42.165,401,6 — 7.529,536 — 53.7824 + 19.2080 — 13.720 — 19.6 — 14 = 76.131,292,16 — 108.631,936; by which it appears that the three terms $x^8 + 4x^7 + 5x^4$, which ought to be greater than all the five terms $x^6 + 10x^5 + 5x^3 + 10x^2 + 10x$, and to exceed them by the number 5, will be leſs than the ſaid five terms. Therefore 1.4 will be leſs than the value of x in the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$.

Art. 47. We will therefore ſuppoſe x to be = 1.6.

And

And then we ſhall have $xx = 2.56$, and $x^3 = 4.096$, and $x^4 = 6.5536$, and $x^5 = 10.48576$, and $x^6 = 16.777,216$, and $x^7 = 26.843,545,6$, and $x^8 = 42.949,672,96$, and conſequently $4x^7$ $(= 4 \times 26.843,545,6) = 107.374,182,4$, and $10x^5$ $(= 10 \times 10.48576) = 104.8576$, and $5x^4$ $(= 5 \times 6.5536) = 32.7680$, and $5x^3$ $(= 5 \times 4.096) = 20.480$, and $10x^2$ $(= 10 \times 2.56) = 25.6$, and $10x$ $(= 10 \times 1.6) = 16$, and $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ $(= 42.949,672,96 + 107.374,182,4 - 16.777,216 - 104.8576 + 32.7680 - 20.480 - 25.6 - 16) = 183.091,855,36 - 183.714,816$; of which quantities the latter, to wit, $183.714,816$, (which is equal to $x^6 + 10x^5 + 5x^3 + 10x^2 + 10x$,) is greater than the former, to wit, $183.091,855,36$, (which is equal to $x^8 + 4x^7 + 5x^4$,) and therefore cannot be ſubtracted from it, as it is ſuppoſed to be in the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$. Therefore the root of that equation muſt be ſomewhat greater than 1.6.

Art. 48. Let us therefore ſuppoſe x to be $= 1.6 + z$, and let us proceed to inveſtigate the value of z by Mr. Raphſon's method of approximation.

Then, ſince x is $= 1.6 + z$, we ſhall have

$$xx \; (= \overline{1.6+z}|^2 = \overline{1.6}|^2 + 2 \times 1.6 \times z + \&c) = 2.56 + 3.2 \times z + \&c,$$

and x^3 $(= \overline{1.6+z}|^3 = \overline{1.6}|^3 + 3 \times \overline{1.6}|^2 \times z + \&c = \overline{1.6}|^3 + 3 \times 2.56 \times z + \&c) = 4.096 + 7.68 \times z + \&c,$

and

and x^4 $(= \overline{1.6+z}|^4 = \overline{1.6}|^4 + 4 \times \overline{1.6}|^3 \times z + \&c$
$= \overline{1.6}|^4 + 4 \times 4.096 \times z + \&c) =$
$6.5536 + 16.384 \times z + \&c,$

and x^5 $(= \overline{1.6+z}|^5 = \overline{1.6}|^5 + 5 \times \overline{1.6}|^4 \times z + \&c$
$= \overline{1.6}|^5 + 5 \times 6.5536 \times z + \&c) =$
$10.48576 + 32.7680 \times z + \&c,$

and x^6 $(= \overline{1.6+z}|^6 = \overline{1.6}|^6 + 6 \times \overline{1.6}|^5 \times z +$
$\&c = \overline{1.6}|^6 + 6 \times 10.48576 \times z + \&c)$
$= 16.777,216 + 62.91456 \times z + \&c,$

and x^7 $(= \overline{1.6+z}|^7 = \overline{1.6}|^7 + 7 \times \overline{1.6}|^6 \times z + \&c$
$= \overline{1.6}|^7 + 7 \times 16.777,216 \times z + \&c)$
$= 26.843,545,6 + 117.440,512 \times z$
$+ \&c,$

and x^8 $(= \overline{1.6+z}|^8 = \overline{1.6}|^8 + 8 \times \overline{1.6}|^7 \times z + \&c$
$= \overline{1.6}|^8 + 8 \times 26.843,545,6 \times z + \&c)$
$= 42.949,672,96 + 214.748,364,8 \times z$
$+ \&c,$

and consequently $4x^7$ $(= 4 \times$
$\overline{26.843,545,6 + 117.440,512 \times z + \&c}$
$= 4 \times 26.843,545,6 + 4 \times 117.440,512$
$\times z + \&c) = 107.374,182,4 + 469.$
$.762,048 \times z + \&c,$

and $10x^5$ ($= 10 \times \overline{10.485,76 + 32.7680 \times z + \&c}$
$= 10 \times 10.485,76 + 10 \times 32.7680$
$\times z + \&c) = 104.8576 + 327.680$
$\times z + \&c$,

and $5x^4$ ($= 5 \times \overline{6.5536 + 16.384 \times z + \&c} = 5 \times$
$6.5536 + 5 \times 16.384 \times z + \&c) =$
$32.7680 + 81.920 \times z + \&c$,

and $5x^3$ ($= 5 \times \overline{4.096 + 7.68 \times z + \&c} = 5 \times$
$4.096 + 5 \times 7.68 \times z + \&c) =$
$20.480 + 38.40 \times z + \&c$,

and $10x^2$ ($= 10 \times \overline{2.56 + 3.2 \times z + \&c} = 10 \times$
$2.56 + 10 \times 3.2 \times z + \&c) = 25.6$
$+ 32 \times z + \&c$,

and $10x$ ($= 10 \times \overline{1.6 + z} = 10 \times 1.6 + 10 \times z$)
$= 16 + 10z$.

Therefore the compound quantity $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ will be equal the complicated quantity

42.949,

$$\left\{\begin{array}{llll}
 & 42.949,672,96 & + \; 214.748,364,8 & \times z + \&c \\
+ & 107.374,182,4 & + \; 469.762,048 & \times z + \&c \\
- & 16.777,216, & - \; 62.914,56 & \times z - \&c \\
- & 104.857,6 & - \; 327.680 & \times z - \&c \\
+ & 32.768,0 & + \; 81.920 & \times z + \&c \\
- & 20.480 & - \; 38.40 & \times z - \&c \\
- & 25.6 & - \; 32 & \times z - \&c \\
- & 16. & - \; 10 & \times z
\end{array}\right\}$$

$$= \left\{\begin{array}{l}
183.091,855,36 + 766,430,412,8 \times z + \&c \\
-183.714,816,00 - 470.994,560,0 \times z - \&c
\end{array}\right\}$$

$$= - \quad 0.622,960,64 + 295.435,852,8 \times z \;\&c.$$

But the compound quantity $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ is $= 5$.

Therefore $295.435,852,8 \times z$ &c $- 0.622,960,64$ will alſo be $= 5$. And conſequently (adding 0.622,960,64 to both ſides,) we ſhall have $295.435,852,8 \times z$ $(= 5 + 0.622,960,64) = 5.622,960,64$, and $z = \frac{5.622,960,64}{295.435,852,8} = 0.019$. Therefore $1.6 + z$, or x, will be $(= 1.6 + 0.019) = 1.619$; that is, the leaſt root of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$ will be 1.619.

Q. E. I.

Art. 49. This value of x will be pretty near the truth, but will exceed it by a very ſmall quantity. For, if we

ſuppoſe

ſuppoſe x to be $= 1.619$, we ſhall have $xx = 2.621,161$, and $x^3 = 4.243,659$, and $x^4 = 6.870,484$, and $x^5 = 11.123,213$, and $x^6 = 18.008,482$, and $x^7 = 29.155,732$, and $x^8 = 47.203,130$, and conſequently $4x^7$ $(= 4 \times 29.155,732) = 116.622,928$, and $10x^5$ $(= 10 \times 11.123,213) = 111.232,130$, and $5x^4$ $(= 5 \times 6.870,484) = 34.352,420$, and $5x^3$ $(= 5 \times 4.243,659) = 21.218,295$, and $10x^2$ $(= 10 \times 2.621,161) = 26.211,610$, and $10x$ $(= 10 \times 1.619) = 16.190,000$, and $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x =$

$$\left\{\begin{array}{rr} 47.203,130 & \\ + 116,622,928 & - 18.008,482 \\ & - 111.232,130 \\ + 34.352,420 & - 21.218,295 \\ & - 26.211,610 \\ & - 16.190,000 \end{array}\right\} =$$

$198.178,478 - 192.860,517 = 5.317,961$; which is ſomewhat greater than 5, or the abſolute term of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$. Therefore 1.619 will be ſomewhat greater than the true value of x in that equation. Q. E. D.

Art. 50. We will therefore, in the next place, ſuppoſe x to be $= 1.619 - v$.

And then we ſhall have

xx (=

xx ($= \overline{1.619 - v}|^2 = \overline{1.619}|^2 - 2 \times 1.619 \times$
$v + \&c = 2.621,161 - 3.238 \times v$
$+ \&c,$

and x^3 ($= \overline{1.619 - v}|^3 = \overline{1.619}|^3 - 3 \times \overline{1.619}|^2 \times$
$v + \&c = \overline{1.619}|^3 - 3 \times 2.621,161$
$\times v + \&c) = 4.243,659 - 7.863,483$
$\times v + \&c,$

and x^4 ($= \overline{1.619 - v}|^4 = \overline{1.619}|^4 - 4 \times \overline{1.619}|^3 \times$
$v + \&c = \overline{1.619}|^4 - 4 \times 4.243,659$
$\times v + \&c) = 6.870,484 - 16.974,636$
$\times v + \&c,$

and x^5 ($= \overline{1.619 - v}|^5 = \overline{1.619}|^5 - 5 \times \overline{1.619}|^4 \times$
$v + \&c = \overline{1.619}|^5 - 5 \times 6.870,484$
$\times v + \&c) = 11.123,213 - 34.352,$
$420 \times v + \&c,$

and x^6 ($= \overline{1.619 - v}|^6 = \overline{1.619}|^6 - 6 \times \overline{1.619}|^5 \times$
$v + \&c = \overline{1.619}|^6 - 6 \times 11.123,213$
$\times v + \&c) = 18.008,482 - 66.739,$
$278 \times v + \&c,$

and x^7 ($= \overline{1.619 - v}|^7 = \overline{1.619}|^7 - 7 \times \overline{1.619}|^6 \times$
$v + \&c = \overline{1.619}|^7 - 7 \times 18,008,482$
$\times v + \&c) = 29.155,732 - 126.059,$
$374 \times v + \&c,$

and

and x^8 $(= \overline{1.619 - v}|^8 = \overline{1.619}|^8 - 8 \times \overline{1.619}|^7 \times v + \&c = \overline{1.619}|^8 - 8 \times 29.155,732 \times v + \&c) = 47.203,130 - 233.245,856 \times v + \&c,$

and consequently $4x^7$ $(= 4 \times 29.155,732 - 4 \times 126.059,374 \times v + \&c) = 116.622,928 - 504.237,496 \times v + \&c,$

and $10x^5$ $(= 10 \times 11.123,213 - 10 \times 34.352,420 \times v + \&c) = 111.232,130 - 343.524,200 \times v + \&c,$

and $5x^4$ $(= 5 \times 6.870,484 - 5 \times 16.974,636 \times v + \&c) = 34.352,420 - 84.873,180 \times v + \&c,$

and $5x^3$ $(= 5 \times 4.243,659 - 5 \times 7.863,483 \times v + \&c) = 21.218,295 - 39.317,415 \times v + \&c,$

and $10x^2$ $(= 10 \times 2.621,161 - 10 \times 3.238 \times v + \&c) = 26.211,610 - 32.380,000 \times v + \&c,$

and $10x$ $(= 10 \times 1.619 - 10 \times v) = 16.190,000 - 10.000,000 \times v,$

and $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x =$

47.203,

$$\left\{\begin{array}{rrrl} & 47.203,130 & - \; 233.245,856 & \times v + \&c \\ + & 116.622,928 & - \; 504.237,496 & \times v + \&c \\ - & 18.008,482 & + \; 66.739,278 & \times v - \&c \\ - & 111.232,130 & + \; 343.524,200 & \times v - \&c \\ + & 34.352,420 & - \; 84.873,180 & \times v + \&c \\ - & 21.2\text{[illegible]}8,295 & + \; 39.317,415 & \times v - \&c \\ - & 26.211,610 & + \; 32.380,000 & \times v - \&c \\ - & 16.190,000 & + \; 10.000,000 & \times v \end{array}\right\}$$

$$= \left\{\begin{array}{rrrl} & 198.178,478 & - \; 822.356,532 & \times v + \&c \\ - & 192.860,517 & + \; 491.960,893 & \times v - \&c \end{array}\right\}$$

$$= \quad 5.317,961 - 330.395,639 \times v \quad \&c.$$

But the compound quantity $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ is $= 5$.

Therefore $5.317,961 - 330.395,639 \times v$ will alſo be $= 5$.

And conſequently $5.317,961$ will be $= 5 + 330.395,639 \times v$, and $0.317,961$ will be $= 330.395,639 \times v$. Therefore v will be $= \dfrac{0.317,961}{330.395,639} = 0.000,962,3$, and $1.619 - v$ will be $(= 1.619,000,0 - 0.000,962,3) = 1.618,037,7$; that is, x, or the leaſt root of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$ will be nearly equal to $1.618,037,7$. Q. E. I.

This

This value, 1.618,037,7, of the least root of the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$ is, probably, exact in the first seven places of figures 1.618,037.

Art. 51. In order to discover whether the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$ has any other root greater than 1.618,037, we may proceed as follows:

Let the root already found, to wit, 1.618,037,7, or (neglecting the three last figures 377,) 1.618, be called a, as before; and the greater root of the equation (if there be any such root,) be called b.

Then we shall have $b^8 + 4b^7 - b^6 - 10b^5 + 5b^4 - 5b^3 - 10b^2 - 10b = 5$, and consequently $= a^8 + 4a^7 - a^6 - 10a^5 + 5a^4 - 5a^3 - 10a^2 - 10a$. Therefore (adding $b^6 + 10b^5 + 5b^3 + 10b^2 + 10b$ to both sides of the equation,) we shall have $b^8 + 4b^7 + 5b^4 = a^8 + 4a^7 - a^6 - 10a^5 + 5a^4 - 5a^3 - 10a^2 - - 10a + b^6 + 10b^5 + 5b^3 + 10b^2 + 10b = a^8 + 4a^7 + b^6 - a^6 + 10b^5 - 10a^5 + 5a^4 + 5b^3 - 5a^3 + 10b^2 - 10a^2 + 10b - 10a$, and (subtracting $a^8 + 4a^7 + 5a^4$ from both sides,) $b^8 - a^8 + 4b^7 - 4a^7 + 5b^4 - 5a^4 = b^6 - a^6 + 10b^5 - 10a^5 + 5b^3 - 5a^3 + 10b^2 - 10a^2 + 10b - 10a$, or $b^8 - a^8 + 4 \times \overline{b^7 - a^7} + 5 \times \overline{b^4 - a^4} = b^6 - a^6 + 10 \times \overline{b^5 - a^5} + 5 \times \overline{b^3 - a^3} + 10 \times \overline{b^2 - a^2} + 10 \times \overline{b - a}$, and (dividing both sides of the equation by $b - a$,) we shall

ſhall have $\frac{b^8 - a^8}{b - a} + 4 \times \frac{b^7 - a^7}{b - a} + 5 \times \frac{b^4 - a^4}{b - a} = \frac{b^6 - a^6}{b - a} + 10 \times \frac{b^5 - a^5}{b - a} + 5 \times \frac{b^3 - a^3}{b - a} + 10 \times \frac{b^2 - a^2}{b - a} + 10 \times \frac{b - a}{b - a}$, or $b^7 + b^6a + b^5a^2 + b^4a^3 + b^3a^4 + b^2a^5 + ba^6 + a^7 + 4 \times \overline{b^6 + b^5a + b^4a^2 + b^3a^3 + b^2a^4 + ba^5 + a^6} + 5 \times \overline{b^3 + b^2a + ba^2 + a^3} = b^5 + b^4a + b^3a^2 + b^2a^3 + ba^4 + a^5 + 10 \times \overline{b^4 + b^3a + b^2a^2 + ba^3 + a^4} + 5 \times \overline{b^2 + ba + a^2} + 10 \times \overline{b + a} + 10$, or $b^7 + ab^6 + a^2b^5 + a^3b^4 + a^4b^3 + a^5b^2 + a^6b + a^7 + 4b^6 + 4ab^5 + 4a^2b^4 + 4a^3b^3 + 4a^4b^2 + 4a^5b + 4a^6 + 5b^3 + 5ab^2 + 5a^2b + 5a^3 = b^5 + ab^4 + a^2b^3 + a^3b^2 + a^4b + a^5 + 10b^4 + 10ab^3 + 10a^2b^2 + 10a^3b + 10a^4 + 5b^2 + 5ab + 5a^2 + 10b + 10a + 10$, or

$$\left\{\begin{array}{rrrrrrrr} b^7 + ab^6 & + a^2b^5 & + a^3b^4 & + a^4b^3 & + a^5b^2 & + a^6b & + a^7 \\ + 4b^6 & + 4ab^5 & + 4a^2b^4 & + 4a^3b^3 & + 4a^4b^2 & + 4a^5b & + 4a^6 \\ & & & + 5b^3 & + 5ab^2 & + 5a^2b & + 5a^3 \end{array}\right\}$$

$$= \left\{\begin{array}{rrrrrr} b^5 & + ab^4 & + a^2b^3 & + a^3b^2 & + a^4b & + a^5 \\ & + 10b^4 & + 10ab^3 & + 10a^2b^2 & + 10a^3b & + 10a^4 \\ & & & + 5b^2 & + 5ab & + 5a^2 \\ & & & & + 10b & + 10a \\ & & & & & + 10, \end{array}\right.$$

and (ſubtracting $a^5 + 10a^4 + 5a^2 + 10a + 10$ from both ſides,)

$b^7 +$

$$\left\{\begin{array}{rrrrrrrr}
b^7 + ab^6 & + a^2b^5 & + a^3b^4 & + a^4b^3 & + a^5b^2 & + a^6b & + a^7 \\
+4b^6 & + 4ab^5 & + 4a^2b^4 & + 4a^3b^3 & + 4a^4b^2 & + 4a^5b & + 4a^6 \\
 & & & + 5b^3 & + 5ab^2 & + 5a^2b & + 5a^3 \\
 & & & & & & - a^5 \\
 & & & & & & - 10a^4 \\
 & & & & & & - 5a^2 \\
 & & & & & & - 10a \\
 & & & & & & - 10
\end{array}\right\}$$

$$= \left\{\begin{array}{rrrrr}
b^5 + & ab^4 + & a^2b^3 + & a^3b^2 + & a^4b \\
 & + 10b^4 & + 10ab^3 & + 10a^2b^2 & + 10a^3b \\
 & & & + 5b^2 & + 5ab \\
 & & & & + 10b,
\end{array}\right\}$$

and (subtracting from both sides of this equation all the terms on the left-hand side of it that involve the powers of b, and placing all the terms that involve the powers of b on the left-hand side of the new equation,)

$$\left\{\begin{array}{lllllll}
 & 10b & + 5b^2 & + 10ab^3 & + 10b^4 & + b^5 & - 4b^6 - b^7 \\
+ & 5ab & + 10a^2b^2 & + a^2b^3 & + ab^4 & - 4ab^5 & - ab^6 \\
+ & 10a^3b & + a^3b^2 & - 5b^3 & - 4a^2b^4 & - a^2b^5 & \\
+ & a^4b & - 5ab^2 & - 4a^3b^3 & - a^3b^4 & & \\
- & 5a^2b & - 4a^4b^2 & - a^4b^3 & & & \\
- & 4a^5b & - a^5b^2 & & & & \\
- & a^6b & & & & &
\end{array}\right\}$$

$$= a^7 + 4a^6 - a^5 - 10a^4 + 5a^3 - 5a^2 - 10a - 10,$$

or (if we range the terms in the several vertical columns on the left-hand side of this equation according to the powers of a,)

10b

$$\left\{\begin{array}{l} \quad 10b + 5b^2 - 5b^3 + 10b^4 + b^5 - 4b^6 - b^7 \\ + 5ab - 5ab^2 + 10ab^3 + ab^4 - 4ab^5 - ab^6 \\ - 5a^2b + 10a^2b^2 + a^2b^3 - 4a^2b^4 - a^2b^5 \\ + 10a^3b + a^3b^2 - 4a^3b^3 - a^3b^4 \\ + a^4b - 4a^4b^2 - a^4b^3 \\ - 4a^5b - a^5b^2 \\ - a^6b \end{array}\right\}$$

$$= a^7 + 4a^6 - a^5 - 10a^4 + 5a^3 - 5a^2 - 10a - 10.$$

But the octinomial quantity $a^7 + 4a^6 - a^5 - 10a^4 + 5a^3 - 5a^2 - 10a - 10$ is $=$

$$\frac{a^8 + 4a^7 - a^6 - 10a^5 + 5a^4 - 5a^3 - 10aa - 10a}{a} =$$

$$\frac{5}{a} = \frac{5}{1.618} = 3.090.$$

Therefore the complicated quantity

$$\left\{\begin{array}{l} \quad 10b + 5b^2 - 5b^3 + 10b^4 + b^5 - 4b^6 - b^7 \\ + 5ab - 5ab^2 + 10ab^3 + ab^4 - 4ab^5 - ab^6 \\ - 5a^2b + 10a^2b^2 + a^2b^3 - 4a^2b^4 - a^2b^5 \\ + 10a^3b + a^3b^2 - 4a^3b^3 - a^3b^4 \\ + a^4b - 4a^4b^2 - a^4b^3 \\ - 4a^5b - a^5b^2 \\ - a^6b \end{array}\right\}$$

will be $= 3.090$.

Art. 52. Now let 1.618 be substituted in the terms of this equation instead of a.

And

And we ſhall have aa $(= \overline{1.618}|^2) = 2.617,924$, and $a^3 = 4.235,801$, and $a^4 = 6.853,526$, and $a^5 = 11.089,005$, and $a^6 = 17.942,010$, and $a^7 = 29.030,172$, and $a^8 = 46.970,818$, and conſequently $5a$ $(= 5 \times 1.618) = 8.090$, and $5a^2$ $(= 5 \times 2.617,924) = 13.089,620$, and $10a^3$ $(= 10 \times 4.235,801) = 42.358,010$, and $4a^5$ $(= 4 \times 11.089,005) = 44.356,020$, and $10b + 5ab - 5a^2b + 10a^3b + a^4b - 4a^5b - a^6b$ $(= 10b + 8.090 \times b - 13.089,620 \times b + 42.358,010 \times b + 6.853,526 \times b - 44.356,020 \times b - 17.942,010 \times b) = 67.301,536 \times b - 75.387,650 \times b$; and $10a^2$ $(= 10 \times 2.617,924) = 26.179,240$, and $4a^4$ $(= 4 \times 6.853,526) = 27.414,104$, and $5b^2 - 5ab^2 + 10a^2b^2 + a^3b^2 - 4a^4b^2 - a^5b^2$ $(= 5b^2 - 8.090 \times b^2 + 26.179,240 \times b^2 + 4.235,801 \times b^2 - 27.414,104 \times b^2 - 11.089,005 \times b^2) = 30.415,041 \times b^2 - 46.593,109 \times b^2$; and $10a$ $(= 10 \times 1.618) = 16.18$, and $4a^3$ $(= 4 \times 4.235,801) = 16.943,204$, and $-5b^3 + 10ab^3 + a^2b^3 - 4a^3b^3 - a^4b^3$ $(= -5b^3 + 16.18 \times b^3 + 2.617,924 \times b^3 - 16.943,204 \times b^3 - 6.853,526 \times b^3) = -28.796,730 \times b^3 + 18.797,924 \times b^3$; and $4a^2$ $(= 4 \times 2.617,924) = 10.471,696$, and $10b^4 + ab^4 - 4a^2b^4 - a^3b^4$ $(= 10b^4 + 1.618 \times b^4 - 10.471,696 \times b^4 - 4.235,801 \times b^4) = 11.618,000 \times b^4 - 14.707,497 \times b^4$; and $4a$ $(= 4 \times 1.618) = 6.472$, and $b^5 - 4ab^5 - a^2b^5$ $(= b^5 - 6.472,000 \times b^5 - 2.617,924 \times b^5) = b^5 - 9.089,924 \times b^5$; and $-4b^6 - ab^6$ $(= -4.000,000 \times b^6 - 1.618,000 \times b^6) = -5.618,000 \times b^6$. Therefore the left-hand ſide of the foregoing equation will be equal to the following compound quantity, to wit,

67.301,

$67.301,536 \times b - 75.387,650 \times b + 30.415,041 \times b^2 - 46.593,109 \times b^2 - 28.796,730 \times b^3 + 18.797,924 \times b^3 + 11.618,000 \times b^4 - 14.707,497 \times b^4 + 1.000,000 \times b^5 - 9.089,924 \times b^5 - 5.618,000 \times b^6 - b^7$. Therefore this laſt compound quantity muſt be equal to 3.090; or, in other words, the five quantities $67.301,536 \times b + 30.415,041 \times b^2 + 18.797,924 \times b^3 + 11.618,000 \times b^4 + 1.000,000 \times b^5$ muſt be greater than the ſeven quantities $75.387,650 \times b + 46.593,109 \times b^2 + 28.796,730 \times b^3 + 14.707,497 \times b^4 + 9.089,924 \times b^5 + 5.618,000 \times b^6 + b^7$, and the exceſs muſt be equal to 3.090. But the ſaid five quantities are, reſpectively, leſs than the five firſt quantities of the ſaid ſeven quantities, and therefore, *à fortiori*, muſt be leſs than all the ſaid ſeven quantities, and therefore the ſaid ſeven quantities cannot be ſubtracted from the ſaid five quantities, as they are ſuppoſed to be in the ſaid laſt equation. And conſequently the ſaid laſt equation is impoſſible. Therefore the ſuppoſition from which the ſaid equation is derived, to wit, the ſuppoſition that the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$ has a ſecond real and affirmative root b, greater than a, or 1.618, muſt be a falſe ſuppoſition. And conſequently we may, at laſt, conclude that 1.618, or 1.618,037, is the only root of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$. Q. E. I.

Art. 53. This root, 1.618, is pretty near the truth. For, if we ſuppoſe x to be $= 1.618$, we ſhall have

x^8

$x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ $(= 46.970,818 + 4 \times 29.030,172 - 17.942,010 - 10 \times 11.089,005 + 5 \times 6.853,526 - 5 \times 4.235,801 - 10 \times 2.617,924 - 10 \times 1.618) = 46.970,818 + 116.120,688 - 17.942,010 - 110.890,050 + 34.267,630 - 21.179,005 - 26.179,240 - 16.180,000 = 197.359,136 - 192.370,305 = 4.988,831$; which is less than 5, or the absolute term of the proposed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$, by only the small difference 0.011, 169, which is less than the 447th part of the absolute term 5. Therefore 1.618 must be somewhat less than, but very nearly equal to, the true value of x in the said equation. Q. E. D.

A SCHOLIUM.

Art. 54. We have now gone through the resolution of three equations consisting of terms marked with the signs — and + alternately, (and which therefore are capable of having more than one real and affirmative root, if the co-efficients of the powers of x and the absolute terms of the equations are of certain relative magnitudes, with respect to each other, that are proper

for

for that purpose,) to wit, the cubick equation $x^3 - 39x^2 + 479x = 1881$, and the equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$, and the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x = 5$, by means of the processes described above in Art. 9, 10, 11, 12, and 13, together with Mr. Raphson's method of approximation, after a tolerably near value of the least root of each of those equations, somewhat less than the true value of it, had been first obtained by those processes. And these processes have been set forth so fully and distinctly that nothing more is necessary, as I conceive, to be added, in order to explain and illustrate them. And I have likewise shewn, at the end of each of these examples, a method of reducing, when such reduction is possible, and of attempting, in other cases, to reduce, the equation (after it's least root, or it's only root, a, has been obtained, to a sufficient degree of exactness, by means of the said processes and of Mr. Raphson's method of approximation,) to an equation one degree lower than the first equation, by putting b for one of it's other roots, which are greater than a, (if it has any such other roots,) and dividing the terms of the equation resulting from that supposition by the residual quantity $b-a$. And by this reduction, or attempt at a reduction, of the said equation to an equation one degree lower, we can usually discover whether the original equation has, or has not, any other root than a. For, if the equation obtained by this division is a possible equation (as was the case in the first of the three foregoing examples, where the equation so obtained was the quadratick equation $30x - xx = 209$, which is a possible equation, and has two roots,) the original equation, from which it was derived, will have more roots

than

than a, to wit, the root, or roots, of the ſaid ſecond equation, which was derived from it by means of the ſaid diviſion: but, if the ſaid ſecond equation appears to be impoſſible (as was the caſe in the ſecond and third of the three foregoing examples,) the original equation, from which it was derived by means of the ſaid diviſion, can have no other root than the root a, which has been already found. And in the former caſe, or when the ſaid ſecond equation is found to be a poſſible one, it's leaſt root may be found by a repetition of the proceſſes above-deſcribed, together with Mr. Raphſon's method of approximation, in the ſame manner as the leaſt root of the original equation had been obtained before. And then the next greater root of the ſaid ſecond equation, and the next greater than that, and ſo on, may be found in the ſame manner, till all it's roots are known; which are likewiſe roots of the original equation from which the ſaid ſecond equation was derived. And thus at laſt all the roots of the ſaid original equation will be obtained: which is all that can be deſired on the ſubject.

Art. 55. But I am apprehenſive that it may be thought that the number of proceſſes to be uſed in this method is too great, ſo as to make the inveſtigation of the value of a, or the leaſt root of the propoſed equation, a tedious and troubleſome buſineſs. That it is ſo, in ſome degree, I readily confeſs: but I apprehend that this difficulty and tedioufneſs are inherent in the ſubject itſelf, and inſeparable from it, and that the leaſt root, or any other root, of an equation of the eighth degree, that has all it's terms compleat (as is the caſe with the equation

equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$,) cannot be found, by any method hitherto discovered, with less trouble than was taken in the foregoing investigation of the root of the said equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$. And it must be observed that the processes by which the near value 1.6 (which was made the basis of the further approximation to the root of the said equation by Mr. Raphson's method of approximation,) was obtained, though they were pretty many in number, were very simple and easy arithmetical operations. For they were only substitutions of the several short numbers 1, 0.7, 0.9, 1.2, 1.3, 1.4, and 1.6, (consisting of only one and two figures,) instead of y, and α, and β, and x, in the compound quantities $y^8 + 4y^7 + 5y^4$, and $5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$, and $5 + \beta^6 + 10\beta^5 + 5\beta^3 + 10\beta^2 + 10\beta$, and $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$. The remaining part of the investigation of the value of a, or of the least root of the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$, after we have obtained 1.6 for a near value of it somewhat less than the truth, is performed by Mr. Raphson's method of approximation, which is substantially the same with Sir Isaac Newton's, and is allowed on all hands to be the shortest and best method that can be taken for the purpose. I therefore conceive that the only answer that can be given to this complaint of the tediousness of the foregoing method of obtaining the least root of an equation of a high degree, is that which is said to have been given by Euclid, the author of the Elements of

Geometry,

Geometry, to Ptolemy Philadelphus, King of Egypt, when he made a like complaint concerning the difficulty of that method of arriving at the knowledge of Geometry, to wit, "*That there is no royal highway to the knowledge of Geometry, and much less to that of the method of resolving an equation of the eighth degree, or of any higher order.*"

End of the Scholium begun in Art. 54.

Art. 56. In the foregoing three equations which we have resolved in the method described above in art. 10, 11, 12, 13, and 14, in order to illustrate the said method, to wit, the cubick equation $x^3 - 39x^2 + 479x = 1881$, and the equation $x^5 - 7x^4 + 20x^3 - 155x^2 = 10{,}000$, and the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$, the highest power of x is marked with the sign $+$, or the sum of the highest power of x and the terms that are added to it is greater than the sum of the other terms of the equation involving the other powers of x, and the latter sum is subtracted from the former sum, and the terms that compose it are consequently marked with the sign $-$. But the method described in art. 10, 11, 12, 13, and 14, is equally useful in investigating the first near value of the least root of an equation that has several roots, when the highest power of x is marked with the sign $-$; of which it may be proper, before I conclude this discourse, to give an example. I shall therefore now proceed to apply this method to the resolution of an equation of this kind; and the equation I shall chuse for this purpose shall be the biquadratick equation $14{,}937x - 1998x^2 + 80x^3$ —

8

— $x^4 = 5000$, which we have already resolved in the former part of this volume, and which resulted from Dr. Wallis's Solution of Colonel Titus's Arithmetical Problem above-mentioned. This equation has been already examined at great length in the former part of this volume in the Appendix to Dr. Halley's Tract; and it has been shewn to have four different real and affirmative roots, to wit, the numbers 0.350,987, &c, 12.756,441, 794,480,744,022, &c, 32.060,290,8, &c, and 34.832, 280, &c, of which the second, or least but one, to wit, 12.756,441,794,480,744,022, &c, was investigated in pages 65, 66, 67, &c - - - 96, and the other three were investigated afterwards in pages 144, 145, 146, &c - - - 175. And in page 161 it is asserted that the first, or least, of these roots, to wit, 0.350,987, &c, would, if computed to a greater degree of exactness, be found to be equal to 0 350,987,045,866,14. This least root I now propose to investigate by the method above-described, so as to find a first near value of it that shall be sufficiently near to it's true value to be made use of as a convenient ground, or basis, of a further approximation to the said true value in the manner recommended by Mr. Raphson; and I shall afterwards, by means of Mr. Raphson's method, continue the investigation of it to twelve or thirteen places of decimal figures, but shall say nothing more of the other three roots of this equation, which have been sufficiently considered in the former part of this volume.

Art. 57. This least root of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ is the more worthy of

our attention and of a more accurate inveſtigation than has been already beſtowed on it, becauſe it is of uſe in the ſolution of Colonel Titus's Problem above-mentioned, as well as the ſecond root 12.756,441,794,480,744,022, &c which Dr. Halley has inveſtigated: for it enables us (as we have ſeen above in pages 234, 235, and 236,) to find the ſecond ſet of numbers, or of values of *a*, *b*, and *c*, in Colonel Titus's Problem, that will anſwer the conditions of that problem, to wit, the numbers 0.418,919,470, &c, 3.912,226,866, &c, and 4.044,884,670, &c. This leaſt root of the ſaid equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ I ſhall therefore now proceed to inveſtigate in the method deſcribed above in art. 10, 11, 12, 13, and 14.

An Inveſtigation of the Leaſt Root of the Biquadratick Equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ *by Means of the Proceſſes deſcribed in Art.* 10, 11, 12, 13, *and* 14, *and Mr. Raphſon's Method of Approximation.*

Art. 58. In the firſt place we will ſeek the root of the curtailed equation $14,937x + 80x^3 = 5000$, or $14,937y + 80y^3 = 5000$, which has but one root. And this one root we will ſeek by conjectures and trials made in the manner directed by Stevinus and Kerſey. This may be done as follows:

Let

Let us firſt ſuppoſe y to be $= 1$, and try the effect of this ſuppoſition.

Now, if y is $= 1$, we ſhall have y^3 alſo $= 1$, and and $80y^3$ $(= 80 \times 1) = 80$. Therefore $14,937y + 80y^3$ will in this caſe be $(= 14,937 + 80) = 15,017$; which is much greater than 5000, or the abſolute term of the equation $14,937y + 80y^3 = 5000$. Therefore 1 muſt be much greater than the true value of y in this equation, or the ſaid true value of y muſt be much leſs than 1.

Secondly, ſince y is leſs than 1, yy will be leſs than y, and y^3 will be leſs than yy, and, *à fortiori*, leſs than y. Therefore $80y^3$ will be much leſs than $14,937y$, and conſequently $14,937y$ alone will be nearly equal to $14,937y + 80y^3$, and therefore to (it's equal,) 5000. Therefore y will be nearly equal to $\frac{5000}{14,937}$, or to 0.33. Therefore, (by the Lemma demonſtrated in art. 10,) 0.33 (which is nearly equal to the ſingle root of the equation $14,937y + 80y^3 = 5000$,) muſt be leſs than the leaſt root of the original equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and may therefore, if it is near enough to the true value of the ſaid leaſt root, be conveniently made uſe of as the ground, or baſis, of a further approximation to the true value of the ſaid leaſt root by putting $x = 0.33 + z$, and ſubſtituting $0.33 + z$ inſtead of x in the terms of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and reſolving the tranſformed equation reſulting from ſuch ſubſtitution as if it

were a mere ſimple equation, according to the directions of Mr. Raphſon.

Art. 59. We muſt therefore in the next place endeavour to diſcover whether 0.33 is near enough to the true value of x, or the leaſt root of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ to be fit to be made uſe of in this manner. And for this purpoſe we muſt ſubſtitute it inſtead of x in the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, and compare the value of the ſaid quadrinomial quantity reſulting from ſuch ſubſtitution with the abſolute term 5000, to which the ſaid quadrinomial quantity is equal in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Now, if x is $= 0.33$, we ſhall have x^2 $(= \overline{0.33}|^2)$ $= 0.1089$, and x^3 $(= \overline{0.33}|^3)$ $= 0.035{,}937$, and x^4 $(= \overline{0.33}|^4)$ $= 0.011{,}859{,}21$, and conſequently $14{,}937x$ $(= 14{,}937 \times 0.33) = 4929.21$, and $1998x^2$ $(= 1998 \times 0.1089) = 217.5822$, and $80x^3$ $(= 80 \times 0.035{,}937)$ $= 2.874{,}960$. Therefore the whole compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be $(= 4929.21 - 217.5822 + 2.874{,}960 - 0.011{,}859{,}21 = 4932.084{,}960 - 217.594{,}059{,}21) = 4714.490{,}900{,}79$; which is conſiderably leſs than 5000, or the abſolute term of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore 0.33 will be conſiderably leſs than the true value of x, or the leaſt root of that equation: and therefore it will be expedient to form another conjecture concerning the ſaid true value of x, that will be rather nearer to it. And, on account of the magnitude of the difference of the foregoing reſult

4714.

4714.490,900,79 from the abfolute term 5000, it feems reafonable to conjecture that the faid true value will be very nearly equal to 0.35. We will therefore fuppofe x to be equal to 0.35, and will fubftitute 0.35 inftead of x in the terms of the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to difcover whether the refult of fuch fubftitution will be greater, or lefs, than the abfolute term 5000, and confequently whether 0.35 will be greater, or lefs, than the true value of x, or the leaft root of the propofed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and likewife whether the faid refult will approach near enough to the faid abfolute term to make it expedient to make ufe of 0.35 as the ground, or bafis, of a further approximation to the true value of the faid leaft root in the manner recommended by Mr. Raphfon.

Art. 60. Now, if x is $= 0.35$, we fhall have x^2 ($= \overline{0.35}|^2$) $= 0.1225$, and x^3 ($= \overline{0.35}|^3$) $= 0.042,875$, and x^4 ($= \overline{0.35}|^4$) $= 0.015,006,25$, and confequently $14,937x$ ($= 14,937 \times 0.35$) $= 5227.95$, and $1998x^2$ ($= 1998 \times 0.1225$) $= 244.7550$, and $80x^3$ ($= 80 \times 0.042,875$) $= 3.430,000$, and confequently $14,937x - 1998x^2 + 80x^3 - x^4$ ($= 5227.95 - 244.7550 + 3.430,000 - 0.015,006,25 = 5231.380,000 - 244.770,006,25$) $= 4986.609,993,75$; which is fomewhat lefs than 5000, or the abfolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$; but the difference between them is but fmall. Therefore 0.35 will be fomewhat lefs than the true value of x, or the faid equation; but the difference between them will be but fmall. Q. E. I.

Art. 61. We will therefore now take 0.35 for the basis of a further approximation towards the true value of x, or the least root of the proposed equation, in the manner recommended by Mr. Raphson, and suppose x to be $=$ 0.35 + z, and substitute the said binomial quantity instead of x in the terms of the said equation. This may be done in the following manner:

Since x is $= 0.35 + z$, we shall have x^2 $(= \overline{0.35 + z}|^2 = \overline{0.35}|^2 + 2 \times 0.35 \times z + \&c = \overline{0.35}|^2 + 0.70 \times z + \&c) = 0.1225 + 0.70 \times z + \&c$,

and x^3 $(= \overline{0.35 + z}|^3 = \overline{0.35}|^3 + 3 \times \overline{0.35}|^2 \times z + \&c = \overline{0.35}|^3 + 3 \times 0.1225 \times z + \&c = \overline{0.35}|^3 + 0.3675 \times z + \&c) = 0.042{,}875 + 0.3675 \times z + \&c$,

and x^4 $(= \overline{0.35 + z}|^4 = \overline{0.35}|^4 + 4 \times \overline{0.35}|^3 \times z + \&c = \overline{0.35}|^4 + 4 \times 0.042{,}875 \times z + \&c = \overline{0.35}|^4 + 0.171{,}500 \times z + \&c) = 0.015{,}006{,}25, + 0.171{,}500 \times z + \&c$.

Therefore $14{,}937x$ will be $(= 14{,}937 \times \overline{0.35 + z} = 14{,}937 \times 0.35 + 14{,}937 \times z) = 5227.95 + 14{,}937z$; and $1998x^2$ will be $(= 1998 \times \overline{0.1225 + 0.70 \times z + \&c} = 1998 \times 0.1225 + 1998 \times 0.70 \times z + \&c) = 244.7550 + 1{,}398.60 \times z$;

and

and $80x^3$ will be $(= 80 \times \overline{0.042,875 + 0.3675 \times z + \&c} = 80 \times 0.042,875 + 80 \times 0.3675 \times z + \&c) = 3.430,000 + 29.4000 \times z + \&c$; and consequently the whole quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be equal to the following compound quantity, to wit,

$$\left\{\begin{array}{llll} & 5227.95 & + & 14,937 \times z \\ - & 244.7550 & - & 1398.60 \times z - \&c \\ + & 3.430,000 & + & 29.4000 \times z + \&c \\ - & 0.015,006,25 & - & 0.171,500 \times z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{llll} & 5231.380,000,00 & + & 14,966.400,000 \times z + \&c \\ - & 244.770,006,25 & - & 1,398.771,500 \times z - \&c \end{array}\right\}$$

$$= 4986.609,993,75 + 13,567.628,500 \times z \ \&c.$$

But the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the quantity $4986.609,993,75 + 13,567.628,500 \times z$ &c will also be $= 5000$. And consequently $13,567.628,500 \times z$ will be $(= 5000.000,000,00 - 4986.609,993,75) = 13.390,006,25$, and z will be $(= \frac{13.390,006,25}{13,567.628,500}) = 0.000,986,9$. Therefore x, or $0.35 + z$, will be $(= 0.35 + 0.000,986,9) = 0.350,986,9$, or, very nearly, $0.350,987$; that is, $0.350,987$ will be another near value of x, or the least root of the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, that will be much nearer than 0.35, or it's last near value, to it's true value. Q. E. I.

Art. 62.

Art. 62. Now, in order to discover how near this last number, 0.350,987, will approach to the true value of x, or the least root of the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and likewise to discover whether it will be greater, or less, than the said true value, let 0.350, 987 be substituted instead of x in the terms of the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$. This may be done as follows:

If x is $= 0.350,987$, we shall have $x^2 = 0.123,191,874,169$, and $x^3 = 0.043,238,746,338,954,803$, and $x^4 = 0.015,176,237,861,270,729,440,561$, and consequently $14,937x$ $(= 14,937 \times 0.350,987) = 5242.692,819$, and $1998x^2$ $(= 1998 \times 0.123,191,874,169) = 246.137,364,589,662$, and $80x^3$ $(= 80 \times 0.043,238,746,338,954,803) = 3.459,099,707,116,384,240$. Therefore the whole quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be $(= 5242.692,819, - 246.137,364,589,662 + 3.459,099,707,116,384,240 - 0.015,176,237,861,270,729,440,561 = 5246.151,918,707,116,384,240 - 246.152,540,827,523,270,729,440,561)$ $= 4999.999,377,879,593,113,510,559,439$; which is less than 5000, or the absolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore 0.350,987 is less than the true value of x, or the least root of that equation. Q. E. I.

Art. 63. We will therefore now put $x = 0.350,987 + z$, and substitute this binomial quantity instead of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and resolve the transformed equation thence arising

arising in the manner prescribed by Mr. Raphson, in order to obtain a still nearer value of x, or the least root of the said equation, than 0.350,987. This may be done in the manner following:

Since x is $= 0.350,987 + z$, we shall have x^2 ($= \overline{0.350,987 + z}|^2 = \overline{0.350,987}|^2 + 2 \times 0.350,987 \times z + \&c = \overline{0.350,987}|^2 + 0.701,974 \times z + \&c$) $= 0.123,191,874,169 + 0.701,974 \times z + \&c$,

and x^3 ($= \overline{0.350,987 + z}|^3 = \overline{0.350,987}|^3 + 3 \times \overline{0.350,987}|^2 \times z + \&c = \overline{0.350,987}|^3 + 3 \times 0.123,191,874,169 \times z + \&c = \overline{0.350,987}|^3 + 0.369,575,622,507 \times z + \&c$) $= 0.043,238,746,338,954,803 + 0.369,575,622,507 \times z + \&c$,

and x^4 ($= \overline{0.350,987 + z}|^4 = \overline{0.350,987}|^4 + 4 \times \overline{0.350,987}|^3 \times z + \&c = \overline{0.350,987}|^4 + 4 \times 0.043,238,746,338,954,803 \times z + \&c = \overline{0.350,987}|^4 + 0.172,954,985,355,819,212 \times z + \&c$) $= 0.015,176,237,861,270,729,440,561 + 0.172,954,985,355,819,212 \times z + \&c$.

Therefore $14,937x$ will be ($= 14,937 \times \overline{0.350,987 + z}| = 14,937 \times 0.350,987 + 14,937 \times z$) $= 5242.692,819 + 14,937z$;

and

and $1998x^2$ will be $(= 1998 \times \overline{0.123,191,874,169 + 0.701,974 \times z + \&c} = 1998 \times 0.123,191,874,169 + 1998 \times 0.701,974 \times z + \&c) = 246.137,364,589,662 + 1402.544,052 \times z + \&c$;

and $80x^3$ will be $(= 80 \times 0.043,238,746,338,954,803 + 80 \times 0.369,575,622,507 \times z + \&c) = 3.459,099,707,116,384,240 + 29.566,049,800,560 \times z + \&c$;

and consequently the whole quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be equal to the following compound quantity, to wit,

$$\left\{\begin{array}{lr} 5242.692,819 & + 14,937 \times z \\ -246.137,364,589,662 & - 1402.544,052 \times z - \&c \\ + \; 3.459,099,707,116,384,240 + & 29.566,049,800,560 \times \\ - \; 0.015,176,237,861,270,729,440,561 & z + \&c \\ & - 0.172,954,985,355,819,212 \times z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{l} 5246.151,918,707,116,384,240 \\ \qquad\qquad + 14,966.566,049,800,560, \times z + \&c \\ -246.152,540,827,523,270,729,440,561 \\ \qquad\qquad - 1402.717,006,985,355,819,212 \times z - \&c \end{array}\right\}$$

$$= 4999.999,377,879,593,113,510,559,439 - 13,563.849,042,815,204,180,788 \times z \; \&c.$$

But the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore

Therefore the quantity 4999.999,377,879,593,113,510, 559,439 + 13,563.849,042,815,204,180,788 × z &c will also be = 5000; and consequently the quantity 13,563.849,042,815,204,180,788 × z &c will be =

$$\left\{ \begin{array}{r} 5000.000,000,000,000,000,000,000,000 \\ -\ 4999.999,377,879,593,113,510,559,439 \end{array} \right\}$$
$$= \quad 0.000,622,120,406,886,489,440,561,$$

and z will be $= \dfrac{0.000,622,120,406,886,489,440,561,}{13,563.849,042,815,204,180,788} =$ 0.000,000,045,866,06 &c. Therefore x, or 0.350,987 + z, will be = 0.350,987 + 0.000,000,045,866,06 &c = 0.350,987,045,866,06 &c. Q. E. I.

Art. 64. This value of x differs from the number 0.350,987,045,866,14 (to which I had asserted in page 161 that the least root of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ would be found to be equal,) in the 13th place of decimal fractions; so that I must have made a slip in one of the two calculations by which I obtained these numbers in that place of figures. But, as this difference of the two numbers occurs only in the 13th figure, we may safely conclude that the first twelve figures 0.350,987,045,866, of the number 0.350, 987,045,866,06, that has just now been obtained by the foregoing investigation, which are the same in both calculations, will be exact.

End of the General Method of investigating the Two or Three First Figures of the Least Root of an Equation that has more than One Real and Affirmative Root.

A SPECIMEN

OF

VIETA's METHOD OF RESOLVING ALGEBRÄICK EQUATIONS OF ANY ORDER BY APPROXIMATION.

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A Specimen of Vieta's *Method of resolving Algebräick Equations of any Order by Approximation; containing an Example of the Resolution of the Equation* $x^5 - 5x^3 + 500x = 7,905,504$, *(which is the Subject of the 15th Problem of his Discourse upon this Subject,) according to his Method.*

Article 1. IN the foregoing part of this Volume of Tracts I have endeavoured to explain and illustrate the three several methods of resolving Affected Algebräick equations by approximation that have been given us by Sir Isaac Newton, Mr. Raphson, and Dr. Halley, and have compared them with each other, in order to enable my readers to judge of their several merits, and determine to which of them they will give the preference. This comparison has been made between Mr. Raphson's and Dr. Halley's methods in the Appendix to Dr. Halley's Tract on this subject, which forms the Second Tract in this collection, and extends from page 25 to page 183. In this Appendix I have resolved each of the three equations which Dr. Halley has brought as

examples

examples of his method of reſolution, to wit, the cubick equation $x^3 - 17x^2 + 54x = 350$, and the biquadratick equations $x^4 - 3x^2 + 75x = 10{,}000$ and $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, in two different manners, to wit, firſt, in the manner preſcribed by Dr. Halley, and, ſecondly, in the manner preſcribed by Mr. Raphſon, that the advantages and diſadvantages of each method may be the more apparent: and I have ſtated my reaſons for thinking that, in moſt caſes, Mr. Raphſon's method deſerves to be preferred to Dr. Halley's. And I have compared Sir Iſaac Newton's method of reſolution with that of Mr. Raphſon in the Tract intitled "*Obſervations on Mr. Raphſon's Method of reſolving Affected Equations of all Degrees by Approximation*," which extends from page 279 to page 323; in which Tract I have reſolved the only equation that has been brought by Sir Iſaac Newton as an example of his method, to wit, the cubick equation $x^3 - 2x = 5$, in two different manners, to wit, firſt, in the manner preſcribed by Sir Iſaac Newton, and, ſecondly, in the manner preſcribed by Mr. Raphſon; and have compared the methods with each other in the ſeveral proceſſes of them, ſtep by ſtep, and mentioned my reaſons for concluding that Mr. Raphſon's method, though it differs but little from Sir Iſaac Newton's, yet deſerves, in ſome degree, to be preferred to it. And, as in all theſe three methods, of Sir Iſaac Newton, Mr. Raphſon, and Dr. Halley, it is preſuppoſed that the calculator is already able to find the value of a, (or the firſt near value of x, or the root ſought,) to a tolerable degree of exactneſs, (or to within one hundredth part, or, at leaſt, to within one tenth part, of the true value

of

of x,) either by conjecture and trial, or otherwise, which first near value, a, is to be made the ground, or basis, of a further approach to the true value of x by their several methods of approximation;—and, as neither of those eminent writers has given his readers any directions as to the manner of finding such first near value;—I have endeavoured to supply this defect by re-printing the 10th Chapter of Mr. John Kersey's excellent Treatise on Algebra, in which that clear and easy writer has set-forth and illustrated by examples the general method given us by Simon Stevinus, (the Flemish Mathematician of the former part of the last century,) for that purpose: and to this method I have subjoined a little improvement of my own, which will be of use in some particular cases, which might otherwise create some doubt and perplexity: I mean those cases in which some of the terms which involve the powers of x in the equation are marked with the sign +, and others of them are marked with the sign −, and in which it is consequently often probable that the equation that is to be resolved may have more than one real and affirmative root; in all which cases the improvement here alluded-to will enable us to find a quantity that will always be less than the least of the several different roots of the equation, if it has more than one such root, or than the only root of it, if, notwithstanding the appearances to the contrary, it should have only one such root; and from this quantity (which is always less than the true value of the said least, or only, root,) we may make gradual approaches, by Stevinus's method, or otherwise, to the said true value, till we are come near enough to it to make it expedient

to proceed afterwards to greater degrees of exactness by Mr. Raphson's method of approximation. And in this last Tract I have likewise shewn how, when the least, or only, root of the proposed equation has been so discovered, we may, if the equation has no other root besides that which has been so discovered, determine with certainty that it has no other root, and, if it has one or more other roots, reduce the equation to another equation one degree lower than the former, which will involve the said other root or roots of the first equation; after which we may, by performing the like processes with respect to the second equation, at last obtain the values of all the several roots of the said first equation, to as great a degree of exactness as we please. And this seems to be all that is necessary to be done towards making this method of resolving equations by approximation, according to the directions given us by Mr. Raphson, quite compleat. I therefore now hope that, after the careful perusal of all the foregoing Tracts contained in this volume, the reader will find himself perfectly master of Mr. Raphson's method of resolving high affected equations by approximation, and will be able to apply it readily to the resolution of any Algebräick equation that may be proposed to him;—to contribute to which was the principal object I had in view in preparing these discourses for the press. I might therefore here with propriety put an end to this volume. But, as there is another method of resolving all equations by approximation, which was invented by the celebrated Vieta, above a hundred years before the publication of Mr. Raphson's and Dr. Halley's methods, and which was generally

generally adopted and practiſed by the ableſt Mathematicians of thoſe times, and particularly by Mr. Oughtred, Dr. Pell, and Dr. Wallis, though it has ſince been abandoned on account of the greater facility of the methods of Sir Iſaac Newton, Mr. Raphſon, and Dr. Halley;—I conceive that it will be agreeable to the readers of theſe Tracts to be made acquainted with the principles on which it is grounded, and the manner in which we are enabled by it to inveſtigate the value of the root of any equation whatſoever. And for this reaſon I ſhall now proceed to lay before the reader a ſpecimen of this method, by reſolving in the manner preſcribed by it one of the affected equations which Vieta himſelf has exhibited as an example of it: but in doing this I ſhall not adopt that writer's language, or phraſeology, becauſe it is now grown ſo obſolete that the uſe of it would only tend to increaſe the difficulty of underſtanding the method itſelf.

Art. 2. The equation I ſhall ſelect for this purpoſe is that which is the ſubject of the 15th Problem of Vieta's Diſcourſe, intitled, "*De Numeroſâ Poteſtatum Adfectarum Reſolutione,*" in pages 208, 209, and 210 of Schooten's edition of his works publiſhed at Leyden in the year 1646, and which is expreſſed as follows in his notation; to wit, $1QC - 5C + 500N = 7,905,504$; that is, "the product of the multiplication of the ſquare (Q) of a certain unknown number (N) into the cube (C) of the ſame number, being firſt diminiſhed by ſubtracting from it 5 times the cube (C) of the ſaid number, and being then increaſed by adding to the remainder of the

said subtraction 500 times the said number (N) itself; and the result being supposed to be equal to the given, or known, number 7,905,504; it is required to find the said number (N) itself." This problem, if expressed in our present more convenient Algebräick notation, will, if we put x for the unknown number sought, be as follows: "If $x^5 - 5x^3 + 500x$ be $= 7,905,504$, it is required to find the number x." We will therefore now endeavour to shew in what manner Vieta proceeds to find the said number x, or the root of the equation $x^5 - 5x^3 + 500x = 7,905,504$.

Art. 3. The trinomial quantity $x^5 - 5x^3 + 500x$, (which forms the first, or left-hand, side of the equation $x^5 - 5x^3 + 500x = 7,905,504$) is called by Vieta *an affected fifth power of x*, in contra-distinction to the single quantity x^5 which he calls *the pure fifth power of x*; and he calls the said trinomial quantity by this name of *an affected fifth power of x*, because the quantity x^5, or the pure fifth power of x, occurs in it, but is intermixed with, and *affected*, or *altered in it's magnitude, by*, the two terms $5x^3$ and $500x$, which involve two lower powers of x, of which the former term $5x^3$ is subtracted from x^5, and therefore tends to diminish it, and the latter term $500x$ is added to x^5, and therefore tends to increase it. And Vieta's manner of investigating the value of x in the said trinomial quantity, or affected fifth power of x, to wit, $x^5 - 5x^3 + 500x$, when the said fifth power is supposed to be equal to the known number 7,905,504, is similar to, and derived from, his manner of investigating the value of x in the pure fifth power of x, to wit, the single

ſingle quantity x^5, when the ſaid pure fifth power of x, or the ſingle quantity x^5, is equal to the ſame known number 7,905,504; or, in other words, his manner of reſolving the affected equation $x^5 - 5x^3 + 500x = 7,905,504$ is ſimilar to, and derived from, his manner of reſolving the pure equation $x^5 = 7,905,504$, or of extracting the fifth root of the known number 7,905,504. Therefore, in order to underſtand Vieta's manner of inveſtigating the value of x in the affected equation $x^5 - 5x^3 + 500x = 7,905,504$, we muſt firſt conſider his manner of inveſtigating the value of x in the pure equation $x^5 = 7,905,504$, or his manner of extracting the 5th root of the number 7,905,504. Now his manner of extracting this 5th root is as follows:

The Extraction of the Fifth Root of the Number 7,905,504, according to Vieta's Method of Inveſtigation.

Art. 4. We muſt firſt compare the given number 7,905,504, (of which we are required to find the 5th root,) with the fifth power of the number 10, in order to determine whether the ſaid 5th root, which we are ſeeking, will be greater, or leſs, than 10.

Now the fifth power of 10 is 100,000, which is much leſs than the given number 7,905,504. Therefore 10 will

will be lefs than the fifth root of the faid given number, or the faid fifth root will be greater than 10.

We muft next compare the given number 7,905,504 with the fifth power of the number 100, in order to determine whether the faid fifth root of 7,905,504, which we are feeking, will be greater, or lefs, than 100.

Now the fifth power of 100 is 10,000,000,000, which is much greater than the given number 7,905,504. Therefore 100 will be greater than the fifth root of the faid given number, or the faid fifth root will be lefs than 100.

We therefore now know that the faid fifth root, which we are feeking, is greater than 10, but lefs than 100.

But the higheft figure of every number that is greater than 10 but lefs than 100 muft be a figure in the place of tens, and muft be either 1, 2, 3, 4, 5, 6, 7, 8, or 9 in the place of tens, and confequently muft be equal to either 10, or 20, or 30, or 40, or 50, or 60, or 70, or 80, or 90.

Therefore the higheft figure of the number which is the fifth root of 7,905,504, will be a figure in the place of tens, and will be either 1, 2, 3, 4, 5, 6, 7, 8, or 9 in the place of tens, and confequently will be equal either to 10, or 20, or 30, or 40, or 50, or 60, or 70, or 80, or 90.

Further,

Further, the number 7,905,504 is greater than 3,200,000, or than 32 × 100,000, or than $\overline{2}|^5 \times \overline{10}|^5$, but lefs than 24,300,000, or than 243 × 100,000, or than $\overline{3}|^5 \times \overline{10}|^5$. Therefore the fifth root of 7,905,504 will be greater than the fifth root of 3,200,000, or than 2 × 10, or than 20; but it will be lefs than the fifth root of 24,300,000, or than 3 × 10, or than 30. Therefore the higheft figure of the number which is the fifth root of 7,905,504 will be a 2 in the place of tens, and will reprefent, or ftand for, 20. And thus we have found the firft, or higheft, figure of the fifth root of 7,905,504, which we are feeking. Q. E. I.

Art. 5. Now let a be put for 20, and z for the unknown remaining part of the fifth root of 7,905,504.

Then will $a + z$ be $= \sqrt[5]{7{,}905{,}504}$, and confequently $\overline{a+z}|^5$ will be $= 7{,}905{,}504$.

But $\overline{a+z}|^5$ is $= a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$.

Therefore $a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7{,}905{,}504$; and (fubtracting a^5 from both fides,) $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7{,}905{,}504 - a^5$ $(= 7{,}905{,}504 - \overline{20}|^5 = 7{,}905{,}504 - 3{,}200{,}000) = 4{,}705{,}504$, which Vieta calls *the Refolvend.* Therefore $5a^4z$ alone will be lefs than 4,705,504, and confequently z will be lefs than $\frac{4{,}705{,}504}{5a^4}$, or than $\frac{4{,}705{,}504}{5 \times \overline{20}|^4}$, or than $\frac{4{,}705{,}504}{5 \times 160{,}000}$,

or

or than $\frac{4,705,504}{800,000}$, or than 5.8 &c. We will therefore suppose z (which we know to be less than 5.8 &c,) to be $= 5$, and will substitute 5 instead of z in the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, in order to discover whether the value of that quinquinomial quantity resulting from such substitution will be equal to, greater than, or less than, 4,705,504, and consequently whether 5 will be equal to, greater than, or less than, the true value of z.

Now, if z is $= 5$, we shall have $z^2 = 25$, and $z^3 = 125$, and $z^4 = 625$, and $z^5 = 3125$, and consequently $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ ($= 5a^4 \times 5 + 10a^3 \times 25 + 10a^2 \times 125 + 5a \times 625 + 3125 = 5 \times 160,000 \times 5 + 10 \times 8000 \times 25 + 10 \times 400 \times 125 + 5 \times 20 \times 625 + 3125 = 800,000 \times 5 + 80,000 \times 25 + 4000 \times 125 + 100 \times 625 + 3125 = 4,000,000 + 2,000,000 + 500,000 + 62,500 + 3125$) $= 6,565,625$. This number is greater than 4,705,504. Therefore 5 must be greater than z, or z (which we before knew to be less than 5.8 &c) will not only be less than 5.8 &c, but also less than 5.

Let us therefore in the next place suppose z to be $= 4$, and try the effect of that supposition.

Now, if z is $= 4$, we shall have $z^2 = 16$, and $z^3 = 64$, and $z^4 = 256$, and $z^5 = 1024$, and consequently $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 5a^4 \times 4 + 10a^3 \times 16 + 10a^2 \times 64 + 5a \times 256 + 1024$ ($= 5 \times$

$\times$ 160,000 $\times$ 4 + 10 $\times$ 8000 $\times$ 16 + 10 $\times$ 400 $\times$ 64 + 5 $\times$ 20 $\times$ 256 + 1024 = 800,000 $\times$ 4 + 80,000 $\times$ 16 + 4000 $\times$ 64 + 100 $\times$ 256 + 1024 = 3,200,000 + 1,280,000 + 256,000 + 25600 + 1024) = 4,762,624. This number is alſo greater than 4,705,504, though not in any great proportion. Therefore 4 muſt be ſomewhat greater than z, or z muſt be leſs than 4, as well as leſs than 5.8 &c and 5. We will therefore in the next place ſuppoſe z to be = 3, and try the effect of that ſuppoſition.

Now, if z is = 3, we ſhall have $z^2 = 9$, and $z^3 = 27$, and $z^4 = 81$, and $z^5 = 243$, and conſequently $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 5a^4 \times 3 + 10a^3 \times 9 + 10a^2 \times 27 + 5a \times 81 + 243$ (= 800,000 $\times$ 3 + 80,000 $\times$ 9 + 4000 $\times$ 27 + 100 $\times$ 81 + 243 = 2,400,000 + 720,000 + 108,000 + 8,100 + 243) = 3,236,343. This number is leſs than 4,705,504. Therefore 3 is leſs than z, and conſequently z (though leſs than 4,) is greater than 3. Therefore the firſt, or higheſt, figure of the value of z will be a 3 in the place of units. Therefore $a + z$ will be greater than a + 3, or 20 + 3, or 23, but leſs than a + 4, or 20 + 4, or 24; and therefore 23 will be the two higheſt figures of the value of $a + z$, or of the fifth root of the propoſed number 7,905,504. Q. E. I.

Art. 6. If more figures of the value of the fifth root of 7,905,504 are wanted, we muſt proceed as follows:

When z is = 3, or $a + z$ is = 23, we have ſeen that the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 +$

$5az^4$

$5az^4 + z^5$ is $= 3,236,343$. But a^5 is $(= \overline{20}|^5) = 3,200,000$. Therefore $a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 3,200,000 + 3,236,343 = 6,436,343$; that is, $\overline{a+z}|^5$, or $\overline{23}|^5$, will be $= 6,436,343$.

Now let a denote 23, inftead of 20; and let z be put for the difference between 23 and the true value of the 5th root of the propofed number 7,905,504.

Then will $\overline{a+z}|^5$ be $= 7,905,504$.

But $\overline{a+z}|^5$ is $= a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$.

Therefore $a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7,905,504$.

But a^5, or $\overline{23}|^5$, is $= 6,436,343$.

Therefore $6,436,343 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7,905,504$; and confequently the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7,905,504 - 6,436,343 = 1,469,161$. Therefore $5a^4z$, or the firft term of the faid quinquinomial quantity, will be lefs than 1,469,161, and confequently z will be lefs than $\frac{1,469,161}{5a^4}$, or than $\frac{1,469,161}{5 \times \overline{23}|^4}$, or than $\frac{1,469,161}{5 \times 279,841}$, or than $\frac{1,469,161}{1,399,205}$, or than 1.04.

We

We will therefore ſuppoſe z to be $= 1$, and will ſubſtitute 1 inſtead of z in the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ (which is equal to 1,469,161,) in order to diſcover whether the reſult will be equal to, greater than, or leſs than, 1,469,161, and conſequently whether 1 will be equal to, greater than, or leſs than, the true value of z in the ſaid equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 1,469,161$, or the true exceſs of the 5th root of 7,905,504 above the number 23.

Now, if z is $= 1$, we ſhall have $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ $(= 5a^4 \times 1 + 10a^3 \times 1 + 10a^2 \times 1 + 5a \times 1 + 1 = 5a^4 + 10a^3 + 10a^2 + 5a + 1 = 5 \times \overline{23}|^4 + 10 \times \overline{23}|^3 + 10 \times \overline{23}|^2 + 5 \times 23 + 1 = 5 \times 279,841 + 10 \times 12,167 + 10 \times 529 + 5 \times 23 + 1 = 1,399,205 + 121,670 + 5290 + 115 + 1) = 1,526,281$. This number is greater than 1,469,161; and conſequently 1 is greater than the true value of z in the ſaid equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 1,469,161$, or than the exceſs of the fifth root of 7,905,504 above the number 23. This alſo might have been concluded from what was ſhewn in art. 5, to wit, that this fifth root would be leſs than 24, though greater than 23.

We will therefore now ſuppoſe z to be $= 0.9$, and will ſubſtitute 0.9 inſtead of z in the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ (which is equal to 1,469,161,) in order to diſcover whether the reſult will be equal to, greater than, or leſs than, 1,469,161, and conſequently whether 0.9 will be equal to, greater

greater than, or lefs than, the true value of z in the faid equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 1,469,161$, or the true excefs of the fifth root of 7,905, 504 above the number 23.

Now, if z is $= 0.9$, we fhall have $z^2 = 0.81$, and $z^3 = 0.729$, and $z^4 = 0.6561$, and $z^5 = 0.590,49$. Therefore $5a^4z$ will be $(= 5a^4 \times 0.9 = 5 \times \overline{23}|^4 \times 0.9 = 5 \times 279,841 \times 0.9 = 1,399,205 \times 0.9) = 1,259,284.5$, and $10a^3z^2$ will be $(= 10 \times a^3 \times 0.81 = 10 \times \overline{23}|^3 \times 0.81 = 10 \times 12,167 \times 0.81 = 121,670 \times 0.81) = 98,552.70$, and $10a^2z^3$ will be $(= 10 \times a^2 \times 0.729 = 10 \times \overline{23}|^2 \times 0.729 = 10 \times 529 \times 0.729 = 5290 \times 0.729) = 3856.410$, and $5az^4$ will be $(= 5 \times 23 \times 0.6561 = 115 \times 0.6561) = 75.4515$, and confequently the whole quinquinomial quantity $5a^4z + + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $(= 1,259,284.5 + 98,552.70 + 3856.410 + 75.4515 + 0.590,49) = 1,361,769.651,99$; which is lefs than $1,469,161$. Therefore 0.9 will be lefs than the true value of z in the equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 1,469,161$, and confequently z will be greater than 0.9, though lefs than 1. Therefore 0.9 will be the firft, or higheft, figure of the true value of z, and confequently 23.9 will be the three firft, or higheft, figures of the value of $a + z$, or of the fifth root of the propofed number 7,905,504. Q. E. I.

Art. 7. If more figures ftill of the true value of the fifth root of 7,905,504 are wanted, we muft now put $a = 23.9$, and $a + z = \sqrt[5]{7,905,504}$, and confequently

quently $\overline{a+z}|^5 = 7,905,504$, or $a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 7,905,504$; whence it will follow that the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7,905,504 - a^5 = 7,905,504 - \overline{23.9}|^5$ ($= 7,905,504 - \overline{23 + 0.9}|^5 = 7,905,504 - \overline{23}|^5 - 5 \times \overline{23}|^4 \times 0.9 - 10 \times \overline{23}|^3 \times 0.81 - 10 \times \overline{23}|^2 \times 0.729 - 5 \times 23 \times 0.9 - 0.590,49 = 7,905,504 - \overline{23}|^5 - 1,361,769 = 7,905,504 - 6,436,343 - 1,361,769.651,99 = 1,469,161 - 1,361,769.651,09) = 107,391.348,01$. Therefore $5a^4z$ alone will be less than $107,391.348,01$, and consequently z will be less than $\frac{107,391.348,01}{5a^4}$, or than $\frac{107,391.348,01}{5 \times \overline{23.9}|^4}$, or than $\frac{107,391.348,01}{5 \times 326,280.8641}$, or than $\frac{107,391.348,01}{1,631,404.3205}$, or than 0.06 &c. Therefore the first, or highest, figure of the true value of z must be either 0.06, that is, 6 in the second place of decimal fractions, or some lower figure; and, if the value of the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ resulting from the substitution of 0.06 in it's terms instead of z, shall be equal to, or less than, $107,391.348,01$, (the absolute term of the last equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 107,391.348,01$,) the said fraction 0.06 will at the same time be either exactly equal to, or less than, the true value of z, and in the latter case the figure 6, in the second place of decimal fractions, will be the first, or highest, figure of the said true value

of

of z. We must therefore now substitute 0.06 instead of z in the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, in order to discover whether the result will be equal to, or less than, 107,391.348,01.

Now, if z is $= 0.06$, we shall have $z^2 = 0.0036$, and $z^3 = 0.000,216$, and $z^4 = 0.000,012,96$, and $z^5 = 0.000,000,777,6$, and consequently $5a^4z$ $(= 5a^4 \times 0.06 = 5 \times \overline{23.9}|^4 \times 0.06 = 5 \times 326,280.8641 \times 0.06 = 1,631,404.3205 \times 0.06) = 97,884.259,230$, and $10a^3z^2$ $(= 10a^3 \times 0.0036 = 10 \times \overline{23.9}|^3 \times 0.0036 = 10 \times 13,651.919 \times 0.0036 = 136,519.19 \times 0.0036) = 491.469,084$, and $10a^2z^3$ $(= 10a^2 \times 0.000,216 = 10 \times \overline{23.9}|^2 \times 0.000,216 = 10 \times 571.21 \times 0.000,216 = 5712.1 \times 0.000,216) = 1.233,813,6$, and $5az^4$ $(= 5a \times 0.000,012,96 = 5 \times 23.9 \times 0.000,012,96 = 119.5 \times 0.000,012,96) = 0.001,548,720$. Therefore the whole quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $(= 97,884.259,230 + 491.469,084 + 1.233,813,6 + 0.001,548,720 + 0.000,000,777,6) = 98,376.963,677,097,6$; which is less than 107,391.348,01. Therefore 0.06 is less than the true value of z in the equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 107,391.348,01$, and will be the first, or highest, figure of the said true value. Therefore the four highest figures of the value of $a + z$, or of $23.9 + z$, or of the fifth root of the proposed number 7,905,504, will be 23.96. Q. E. I.

And in this manner we may continue the investigation of this fifth root to as many decimal figures as we please, by

by putting a for the part of the root that is already discovered, and x for the unknown remainder of it, and proceeding in the manner that has been described in the foregoing articles; every new process of the computation giving us a new figure of the said root.

A SCHOLIUM.

Art. 8. THIS method of extracting the fifth root of the number 7,905,504 is perfectly just and accurate, but is, as we have seen, attended with a great deal of labour. And this has induced the Mathematicians of the latter part of the last century to look-out for easier methods of obtaining the same end. And in this they have been remarkably successfull, partly by means of the noble invention of Logarithms, (found out by Lord Napier in the year 1614, and brought to great perfection by Mr. Henry Briggs in the year 1624,) and partly by Mr. Raphson's and Monsieur de Lagny's methods of extracting the roots of numbers by approximation. For by the common Tables of Logarithms we may find any root of a given number exact to five places of figures with very little trouble, without making use of the proportional parts that are set down in these Tables; and by the help of those proportional parts we may obtain these roots to two figures more, that is, to seven places of figures, or, at least, to six places of figures. And, when we have by this, or any other, means found the root of a number exact to any number of figures, we

may, by only one application of the expreſſion given us by *Monſieur de Lagny* for finding a more exact value of it, find twice as many more figures, or, at leaſt, twice as many more figures wanting one, of the ſaid root, as we had before obtained by means of the Table of Logarithms, or in any other way; ſo that, if the root already found (by means of the Table of Logarithms, or otherwiſe) was exact to five places of figures, we may find, by that expreſſion of Monſieur de Lagny, ten more figures, or, at leaſt, nine more figures, of the ſaid root, that will be all exact. Of this I will now give an example in the caſe of the fifth root of the number 7,905,504, which we have juſt now extracted by the method of Vieta, and of which we have found the firſt four figures to be 23.96.

The Extraction of the Fifth Root of the ſame Number, 7,905,504, by Means of the Expreſſion given by Monſieur de Lagny for that Purpoſe.

Art. 9. Monſieur de Lagny's general expreſſion for the value of any root of a given number, when a firſt near value of the ſaid root, that is leſs than it's true value, has been already found, is as follows: If the given number be called N, and the *m*th root of the ſaid number be required to be found, *m* being any whole number whatſoever; and *a* be a firſt near value of the root ſought, that is leſs than it's true value; the more accurate value of the ſaid *m*th root of the number N will be very nearly equal to $a + \dfrac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$.

Now

Now let it be required to find the fifth root of the number 7,905,504. And let it be ſuppoſed that we have already found, (either by a Table of Logarithms, or in ſome other way,) that the three firſt figures of this fifth root are 23.9.

Then will N be = 7,905,504, and m will be = 5, and a will be = 23.9, and a^m will be $(= \overline{23.9}|^5) =$ 7,798,112.651,99, and $N - a^m$ will be (= 7,905,504 − 7,798,112.651,99) = 107,391.348,01, and $2a$ will be (= 2 × 23.9) = 47.8, and $2a \times \overline{N - a^m}$ will be (= 47.8 × 107,391.348,01) = 5,133,306.434,878. And $m - 1$ will be (= 5 − 1) = 4, and $m + 1$ will be (= 5 + 1) = 6, and $\overline{m-1}| \times N$ will be (= 4 × 7,905,504) = 31,622,016, and $\overline{m+1}| \times a^m$ will be (= 6 × 7,798,112.651,99) = 46,788,675.911,94, and $\overline{m-1}| \times N + \overline{m+1}| \times a^m$ will be (= 31,622,016 + 46,788,675.911,94) = 78,410,691.911,94. Therefore

$$\frac{2a \times \overline{N - a^m}}{\overline{m-1}| \times N + \overline{m+1}| \times a^m} \text{ will be } = \frac{5,133,306.434,878}{78,410,691.911,94}$$

$$= 0.065,466,9. \text{ Therefore } a + \frac{2a \times \overline{N - a^m}}{\overline{m-1}| \times N + \overline{m+1}| \times a^m}$$

will be = 23.9 + 0.065,466,9, or 23.965,466,9; that is, the fifth root of the number 7,905,504 will be very nearly = 23.965,466,9. Q. E. I.

The Extraction of the Fifth Root of the same Number, 7,905,504, by Means of the Expression given by Mr. Raphson for that Purpose.

Art. 10. Mr. Raphson's expression for the second near value of the *m*th root of the number N is $a + \frac{N - a^m}{ma^{m-1}}$; which in the present case is $= 23.9 + \frac{7{,}905{,}504 - \overline{23.9}|^5}{5 \times \overline{23.9}|^4}$ $(= 23.9 + \frac{7{,}905{,}504 - 7{,}798{,}112.651{,}99}{5 \times 326{,}280.8641} = 23.9 + \frac{107{,}391.348{,}01}{1{,}631{,}404.3205} = 23.9 + 0.0658) = 23.9658$. Therefore, according to this expression given us by Mr. Raphson, the 5th root of the number 7,905,504 is $=$ 23.9658. Q. E. I.

This value of $\sqrt{5}\overline{|7{,}905{,}504}$ agrees with the former value of it found above by Monsieur de Lagny's more accurate expression, to wit, 23.965,466,9, in the five first figures 23.965, which therefore must be exact: and consequently this simple expression of Mr. Raphson has given us two new figures of the root sought, to wit, the two figures 0.065, exactly.

Of the Resemblance between Vieta's Method of extracting the said Fifth Root of 7,905,504, and Mr. Raphson's Method of extracting it.

Art. 11. Though, according to Vieta's method of proceeding in the extraction of these roots, we obtain but

but one new figure of the root by every new procefs of the inveftigation (which makes the extraction of a root to ten or twelve places of figures by this method intolerably laborious,) yet, in truth, each procefs of the inveftigation would, if we continued the divifion of the known number that is equal to the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, or to $7{,}905{,}504 - a^5$, (which known number Vieta calls *the refolvend*,) to more than one figure in the quotient, give us, at leaft, as many new figures of the root, wanting one figure, exact as there are figures in a, or the part of the root that is already known. Thus, for example, in the laft procefs of the foregoing inveftigation, if we had continued the divifion of the laft dividend, or *refolvend*, 107,391.348,01 by the divifor $5a^4$, or 1,631,404,3205, (in art. 7,) to two figures in the quotient, (which would have been the two figures 0.065,) we fhould have obtained the fifth figure of the true value of $a + z$, or of the 5th root of 7,905,504, to wit, 0.005, as well as the fourth figure of it, to wit, 0.06, and we might have concluded that 23.965 was the value of the faid fifth root to five places of figures, inftead of concluding that 23.96 was it's value to only four places of figures. And this obfervation may, perhaps, have been the ground of Mr. Raphfon's very fimple and ufefull method of extracting the roots of numbers by approximation, (which I have fet forth and explained in pages 508, 509, and 510 of a Tract publifhed in the year 1795 in a large volume in octavo, containing Mr. James Bernoulli's Doctrine of Permutations and Combinations, and other ufefull tracts on mathematical fubjects,) the very fame operation of divifion, with the fame dividend and divifor, which is

here performed in one of the proceſſes of Vieta's method of extracting the roots of numbers, being performed alſo in Mr. Raphſon's method of extracting them, but with a continuation of the diviſion to more than one figure in the quotient, and, in general, to ſo many figures, wanting one, as there are figures in a, or the part of the root that is already known. But in the foregoing inveſtigation of the fifth root of the number 7,905,504, (contained in art. 4, 5, 6, and 7,) I have adhered ſtrictly to Vieta's method of proceeding, by which only one new figure of the root is obtained by each new proceſs of the inveſtigation; becauſe the only reaſon for my exhibiting in the foregoing articles this example of Vieta's method of extracting the roots of numbers, or reſolving equations involving only pure powers of an unknown quantity, was to lay a foundation for the more ready apprehenſion of his method of proceeding in the extraction of the roots of affected powers of an unknown quantity, or in the reſolution of affected equations. I therefore ſhall now endeavour to explain his method of reſolving the above-mentioned affected equation of the fifth order, to wit, $x^5 - 5x^3 + 500x = 7{,}905{,}504$.

A Reſolution of the Affected Equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$ *in the Manner preſcribed by Vieta.*

Art. 12. Vieta begins his reſolution of this affected equation by obſerving that it's root muſt be nearly equal to the root of the pure equation $x^5 = 7{,}905{,}504$, which has the ſame known quantity, or abſolute term 7,905,

504,

504, as the ſaid affected equation: and he thence concludes that the firſt, or higheſt, figure of the value of x in the ſaid affected equation will be a figure in the place of tens, and will be a 2, and conſequently will be equal to 20; becauſe the firſt, or higheſt, figure of the value of x in the pure equation $x^5 = 7,905,504$, or the firſt, or higheſt, figure of the fifth root of the number 7,905,504, is a figure in the place of tens and is a 2, and conſequently is equal to 20; as we have ſeen in the foregoing inveſtigation of it. Vieta then ſubſtitutes 20 inſtead of x in the trinomial quantity $x^5 - 5x^3 + 500x$ (which he calls the fifth power of x affected by the ſubtraction of 5 times the cube of x from it, and by the addition of 500 times the firſt, or ſimple, power of x, or 500 times the unknown quantity x itſelf, to it,) in order to diſcover whether the reſult of ſuch ſubſtitution will be equal to, greater than, or leſs than, the abſolute term 7,905,504.

Now, if x be ſuppoſed to be $= 20$, we ſhall have $x^3 = 8000$, and $x^5 = 3,200,000$, and $5x^3$ $(= 5 \times 8000) = 40,000$, and $500x$ $(= 500 \times 20) = 10,000$. Therefore $x^5 - 5x^3 + 500x$ will be $(= 3,200,000 - 40,000 + 10,000 = 3,200,000 - 30,000) = 3,170,000$, which is leſs than 7,905,504, or the abſolute term of the propoſed equation $x^5 - 5x^3 + 500x = 7,905,504$, or than the true value of the trinomial quantity $x^5 - 5x^3 + 500x$ in that equation. And hence he concludes that 20 muſt be leſs than the true value of x in that equation, and that the ſaid true value will be ſome number greater than 20, but leſs than 30.

Art. 13.

Art. 13. This conclusion will be confirmed by, secondly, supposing x to be $= 30$, and substituting 30 instead of x in the trinomial quantity $x^5 - 5x^3 + 500x$. For, if x is $= 30$, we shall have $x^3 = 27{,}000$, and $x^5 = 24{,}300{,}000$, and $5x^3$ $(= 5 \times 27{,}000) = 135{,}000$, and $500x$ $(= 500 \times 30) = 15{,}000$, and consequently $x^5 - 5x^3 + 500x$ $(= 24{,}300{,}000 - 135{,}000 + 15{,}000 = 24{,}300{,}000 - 120{,}000) = 24{,}180{,}000$; which is much greater than 7,905,504. Therefore, while x increases from 20 to 30, the trinomial quantity $x^5 - 5x^3 + 500x$ will increase from 3,170,000 to 24,180,000; and consequently there must have been a point of time, during the increase of x from 20 to 30, at which the said trinomial quantity will have been exactly equal to any quantity greater than 3,170,000 and less than 24,180,000, and consequently to the quantity 7,905,504, or the absolute term of the equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$; or there will be some quantity greater than 20, but less than 30, that, being substituted instead of x in the trinomial quantity $x^5 - 5x^3 + 500x$, will make the said trinomial quantity be exactly equal to 7,905,504, or the absolute term of the proposed equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$; or, in other words, the root of the said equation will be greater than 20, but less than 30. Q. E. D.

Art. 14. Having thus found that x, or the root of the equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$, is greater than 20, but less than 30, Vieta proceeds to investigate the addition that must be made to 20 in order to make it equal

equal to x, or the root of the said equation. And his method of making this investigation is in substance equivalent to the following train of reasoning, though expressed in other words.

Let the letter a be put for 20, or the part of the root x that is already known; and let z be put for the unknown remainder of the said root.

Then will x be $= a + z$, and consequently x^5 will be $(= \overline{a+z}|^5) = a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, and x^3 will be $(= \overline{a+z}|^3) = a^3 + 3a^2z + 3az^2 + z^3$, and consequently $5x^3$ will be $(= 5 \times \overline{a^3 + 3a^2z + 3az^2 + z^3}) = 5a^3 + 15a^2z + 15az^2 + 5z^3$, and $500x$ will be $(= 500 \times \overline{a+z}) = 500a + 500z$. Therefore the trinomial quantity $x^5 - 5x^3 + 500x$ will be = the multinomial quantity

$$\left\{\begin{array}{l} \quad a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 \\ - \; 5a^3 - 15a^2z - 15az^2 - 5z^3 \\ + \; 500a + 500z. \end{array}\right\}$$

But the trinomial quantity $x^5 - 5x^3 + 500x$ is = 7,905,504.

Therefore the multinomial quantity

$$\left\{\begin{array}{l} \quad a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 \\ - \; 5a^3 - 15a^2z - 15az^2 - 5z^3 \\ + \; 500a + 500z \end{array}\right\}$$

will also be = 7,905,504.

But

But $a^5 - 5a^3 + 500a$ has been ſhewn to be $=$ 3,170,000; and $5a^4$ is $(= 5 \times \overline{20}|^4 = 5 \times 160,000) = 800,000$, and $15a^2$ is $(= 15 \times 400) = 6000$, and conſequently $5a^4 - 15a^2 + 500$ is $(= 800,000 - 6000 + 500 = 794,000 + 500) = 794,500$; and $10a^3$ is $(= 10 \times 8000) = 80,000$, and $15a$ is $(= 15 \times 20) = 300$, and conſequently $10a^3 - 15a$ is $(= 80,000 - 300) = 79,700$; and $10a^2$ is $(= 10 \times 400) = 4000$, and conſequently $10a^2 - 5$ is $(= 4000 - 5) = 3995$; and $5a$ is $(= 5 \times 20) = 100$.

Therefore the multinomial quantity

$$\left\{\begin{array}{llllll} a^5 + & 5a^4z + & 10a^3z^2 + & 10a^2z^3 + & 5az^4 + & z^5 \\ - & 5a^3 - & 15a^2z - & 15az^2 - & 5z^3 & \\ + & 500a + & 500z & & & \end{array}\right\}$$

is $=$ the ſextinomial quantity $3,170,000 + 794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$.

Therefore the ſextinomial quantity $3,170,000 + 794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$ will be $= 7,905,504$. And conſequently (ſubtracting 3,170,000 from both ſides,) the quinquinomial quantity $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$ will be $(= 7,905,504 - 3,170,000) = 4,735,504$.

Therefore the ſingle term $794,500z$ will be leſs than 4,735,504, and conſequently z will be leſs than $\frac{4,735,504}{794,500}$, or than 5.96 &c. We will therefore ſup-

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pofe z to be $= 5$, and will fubftitute 5 inftead of z in the quinquinomial quantity $794{,}500z + 79{,}700z^2 + 3995z^3 + 100z^4 + z^5$, in order to difcover whether the value of the faid quinquinomial quantity refulting from this fubftitution will be equal to, greater than, or lefs than, 4,735,504, or the abfolute term of the laft equation $794{,}500z + 79{,}700z^2 + 3995z^3 + 100z^4 + z^5 = 4{,}735{,}504$, and confequently whether 5 will be equal to, greater than, or lefs than, the true value of z in that equation.

Now, if z is $= 5$, we fhall have $z^2 = 25$, and $z^3 = 125$, and $z^4 = 625$, and $z^5 = 3125$, and confequently $794{,}500z$ $(= 794{,}500 \times 5) = 3{,}972{,}500$, and $79{,}700z^2$ $(= 79{,}700 \times 25) = 1{,}992{,}500$. Therefore $794{,}500z + 79{,}700z^2$ will be $(= 3{,}972{,}500 + 1{,}992{,}500) = 5{,}965{,}000$, which is greater than 4,735,404. Therefore, *à fortiori*, the whole quinquinomial quantity $794{,}500z + 79{,}700z^2 + 3995z^3 + 100z^4 + z^5$ muft be greater than 4,735,504; and confequently 5 muft be greater than the true value of z in the faid equation $794{,}500z + 79{,}700z^2 + 3995z^3 + 100z^4 + z^5 = 4{,}735{,}504$.

Q. E. I.

We will therefore in the next place fuppofe z to be $= 4$, and will fubftitute 4 inftead of z in the terms of the faid quinquinomial quantity $794{,}500z + 79{,}700z^2 + 3995z^3 + 100z^4 + z^5$, in order to difcover whether the value of the faid quinquinomial quantity refulting from fuch fubftitution will be equal to, greater than, or

lefs

leſs than, the number 4,735,504, or the abſolute term of the equation $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5 = 4,735,504$, and conſequently whether 4 is equal to, greater than, or leſs than, the true value of z in that equation.

Now if z is $= 4$, we ſhall have $z^2 = 16$, and $z^3 = 64$, and $z^4 = 256$, and $z^5 = 1024$. Therefore $794,500z$ will be $(= 794,500 \times 4) = 3,178,000$, and $79,700z^2$ will be $(= 79,700 \times 16) = 1,275,200$, and $3995z^3$ will be $(= 3995 \times 64) = 255,680$, and $100z^4$ will be $(= 100 \times 256) = 25,600$, and conſequently the whole quinquinomial quantity $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$ will be $(= 3,178,000 + 1,275,200 + 255,680 + 25,600 + 1024) = 4,735,504$. Therefore 4 is exactly equal to the true value of z in the equation $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5 = 4,735,504$; and conſequently $a + z$, or $20 + z$, is exactly equal to $20 + 4$, or 24, or the root x of the propoſed equation $x^5 - 5x^3 + 500x = 7,905,504$ is exactly equal to 24. Q. E. I.

Art. 15. And, accordingly, if we ſuppoſe x to be equal to 24, we ſhall have xx $(= \overline{24}|^2) = 576$, and x^3 $(= \overline{24}|^3) = 13,824$, and x^4 $(= \overline{24}|^4) = 331,776$, and x^5 $(= \overline{24}|^5) = 7,962,624$, and $5x^3$ $(= 5 \times 13,824) = 69,120$, and $500x$ $(= 500 \times 24) = 12,000$, and conſequently $x^5 - 5x^3 + 500x$ $(= 7,962,624 - 69,120 + 12,000 = 7,974,624 - 69,120) = 7,905,504$.

504. Therefore 24 is the true value of x in the proposed equation $x^5 - 5x^3 + 500x = 7,905,504$.

Q. E. D.

A Resolution of the same Equation $x^5 - 5x^3 + 500x = 7,905,504$ *by Mr. Raphson's Method of Approximation.*

Art. 16. I will now resolve the foregoing equation $x^5 - 5x^3 + 500x = 7,905,504$ by Mr. Raphson's method of approximation, that the circumstances in which it agrees with Vieta's method of resolving it, and those in which the methods differ from each other, may be the more apparent. And, in order to obtain the value of a, or a first near value of the root x, to be made the ground, or basis, of a further approach to it's true value in the manner prescribed by Mr. Raphson, it will be convenient to proceed as follows:

Let us, first, suppose x to be $= 10$, and try the effect of that supposition.

Now, if x is $= 10$, we shall have $x^3 = 1000$, and $x^5 = 100,000$, and $5x^3 = 5000$, and $500x$ $(= 500 \times 10) = 5000$, and consequently $x^5 - 5x^3 + 500x$ $(= 100,000 - 5000 + 5000) = 100,000$; which is much less

lefs than 7,905,504, or the abfolute term of the propofed equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$.

We will therefore, in the fecond place, fuppofe x to be $= 20$, and try the effect of this fuppofition.

Now, if x is $= 20$, we fhall have $x^3 = 8000$, and $x^5 = 3{,}200{,}000$ and $5x^3$ $(= 5 \times 8000) = 40{,}000$, and $500x$ $(= 500 \times 20) = 10{,}000$, and confequently $x^5 - 5x^3 + 500x = 3{,}200{,}000 - 40{,}000 + 10{,}000 = 3{,}210{,}000 - 40{,}000 = 3{,}170{,}000$; which is alfo lefs than 7,905,504, or the abfolute term of the propofed equation.

We will therefore, in the third place, fuppofe x to be $= 30$, and try the effect of that fuppofition.

Now, if x is $= 30$, we fhall have $x^3 = 27{,}000$, and $x^5 = 24{,}300{,}000$, and $5x^3$ $(= 5 \times 27{,}000) = 135{,}000$, and $500x$ $(= 500 \times 30) = 15{,}000$, and confequently $x^5 - 5x^3 + 500x$ $(= 24{,}300{,}000 - 135{,}000 + 15{,}000 = 24{,}315{,}000 - 135{,}000) = 24{,}180{,}000$; which is greater than 7,905,504, or the abfolute term of the propofed equation.

Therefore, while x increafes from 20 to 30, the trinomial quantity $x^5 - 5x^3 + 500x$ will increafe from 3,170,000 to 24,180,000, and confequently will, at fome one inftant of time during the faid increafe of x from 20 to 30, be equal to any quantity that is greater than 3,170,000 and lefs than 24,180,000, and therefore to the quantity

quantity 7,905,504, or the abſolute term of the propoſed equation $x^5 - 5x^3 + 500x = 7,905,504$; or, there will be ſome quantity greater than 20, but leſs than 30, that, being ſubſtituted inſtead of x in the trinomial quantity $x^5 - 5x^3 + 500x$, will make the ſaid quantity be equal to 7,905,504; that is, the root of the propoſed equation $x^5 - 5x^3 + 500x = 7,905,504$ will be greater than 20, but leſs than 30.

Art. 17. We might now proceed to make further conjectures and trials concerning the magnitude of x, in order to approach a little nearer to it's true value than we have hitherto done, before we have recourſe to Mr. Raphſon's method of approximation to obtain a more exact value of it. But, as in Vieta's method of reſolving this equation, (which has been deſcribed in the preceeding articles,) the ſecond proceſs of the inveſtigation begins from a ſuppoſition that by the firſt proceſs of it we have only diſcovered that x is greater than 20, I will take 20 for the value of a, or the firſt near value of x, from which I will begin the further approach to it's true value by Mr. Raphſon's method; by which means the ſimilitude of the two methods to each other, and likewiſe the points in which they differ from each other, will be ſeen more diſtinctly.

I will therefore now ſuppoſe z to be equal to the exceſs of the true value of x above 20, or a, it's near value, and ſubſtitute $20 + z$, or $a + z$, inſtead of x, in the propoſed equation $x^5 - 5x^3 + 500x = 7,905,504$, omitting all the terms that involve any higher

powers of z than it's simple power, or z itself, agreeably to Mr. Raphson's directions.

Now, if x is $= a + z$, we shall have $x^3 = \overline{a + z}|^3$ $= a^3 + 3a^2z + \&c$, and $x^5 = \overline{a + z}|^5 = a^5 + 5a^4z + \&c$, and $5x^3$ $(= 5 \times \overline{a^3 + 3a^2z + \&c}) = 5a^3 + 15a^2z + \&c$, and $500x$ $(= 500 \times \overline{a + z}) = 500a + 500z$, and consequently $x^5 - 5x^3 + 500x =$

$$\left\{\begin{array}{l} a^5 + 5a^4z + \&c \\ - 5a^3 - 15a^2z - \&c \\ + 500a + 500z \end{array}\right\} =$$

$$\left\{\begin{array}{l} 3,200,000 + 5 \times 160,000 \times z + \&c \\ - 5 \times 8000 - 15 \times 400 \times z - \&c \\ + 500 \times 20 + 500z \end{array}\right\} =$$

$$\left\{\begin{array}{l} 3,200,000 + 800,000 \times z + \&c \\ - 40,000 - 6000 \times z - \&c \\ + 10,000 + 500 \times z \end{array}\right\} =$$

$$\left\{\begin{array}{l} 3,210,000 + 800,500 \times z + \&c \\ - 40,000 - 6000 \times z - \&c \end{array}\right\}$$

$= 3,170,000 + 794,500 \times z$ &c.

But $x^5 - 5x^3 + 500x$ is $= 7,905,504$.

Therefore $3,170,000 + 794,500 \times z$ &c will also be $= 7,905,504$; and consequently $794,500 \times z$ &c will be $(= 7,905,504 - 3,170,000) = 4,735,504$; that is,

794,500

794,500 × z together with the omitted terms included under the &c, will be = 4,735,504. Therefore 794,500 × z alone will be somewhat less than, but nearly equal to, 4,735,504; and consequently z will be somewhat less than, but nearly equal to, $\frac{4,735,504}{794,500}$, or 5.96 &c. Therefore $a + z$, or $20 + z$, or the root of the proposed equation $x^5 - 5x^3 + 500x = 7,905,504$, will be somewhat less than, but nearly equal to, 20 + 5.96 &c, or 25.96 &c.

We will therefore suppose z to be equal to 5, and consequently x, or $20 + z$, to be equal to 25, and try the effect of that supposition.

Now, if x is = 25, we shall have x^3 $(= \overline{25}|^3)$ = 15,625, and x^5 = 9,765,625, and $5x^3$ (= 5 × 15,625) = 78,125, and $500x$ (= 500 × 25) = 12,500, and consequently $x^5 - 5x^3 + 500x$ (= 9,765,625 − 78,125 + 12,500 = 9,778,125 − 78,125) = 9,700,000; which is considerably greater than 7,905,504, or the absolute term of the equation $x^5 - 5x^3 + 500x = 7,905,504$. Therefore 25 is greater than the true value of x in that equation.

We will therefore, in the next place, suppose z to be = 4, and consequently $20 + z$, or x, to be = 20 + 4, or 24, and try the effect of that supposition.

Now, if x is = 24, we shall have x^3 $(= \overline{24}|^3)$ = 13,824, and x^5 $(= \overline{24}|^5)$ = 7,962,624, and $5x^3$ (= 5 ×

$\times$ 13,824) = 69,120, and $500x$ (= 500 $\times$ 24) = 12,000, and consequently $x^5 - 5x^3 + 500x$ (= 7,962,624 — 69,120 + 12,000 = 7,974,624 — 69,120) = 7,905,504, or the absolute term of the proposed equation $x^5 - 5x^3 + 500x = 7,905,504$. Therefore 24 is the true value of x in the said equation.

Q. E. I.

A SCHOLIUM.

Art. 18. We may observe that in this application of Mr. Raphson's method of approximation to the finding a second near value of x after having already found that it was greater than 20, we came to the division of the same dividend 4,735,504, by the same divisor 794,500, by means of which we had obtained the value of z, or the difference between 20 and the true value of x in Vieta's investigation; so that the reasonings used in the two methods in this part of the investigation are the same in substance, and produce the same final result, namely, a near value of z equal to the quotient, 5.96, of the division of the number 4,735,504 by the number 794,500. But in Mr. Raphson's method of proceeding the four latter terms of the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, or $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$, are omitted, as unimportant to the

the final refult, which is obtained by fuppofing the firft of thofe terms alone, to wit, the term $5a^4z$, or $794,500z$, to be equal to the number 4,735,504, and dividing the faid number (which Vieta calls *the refolvend*, by $5a^4$, or 794,500, or the co-efficient of z. This omiffion is a faving of unneceffary labour in Mr. Raphfon's method of proceeding, and feems to be the only advantage it has over Vieta's method in the refolution of the foregoing equation: But, when a, or the known part of the value of x in any equation, confifts of three, or four, or more, figures that are all true, or exact, Mr. Raphfon's method has a great advantage over Vieta's, which is owing to his continuing the quotient arifing from the divifion of the refolvend by $5a^4$, or the co-efficient of z, (which divifion occurs both in Vieta's method and in Mr. Raphfon's,) to more figures than Vieta does, namely, to as many figures, wanting one, as there are figures in a, or the part of the value of x that is already known; all which figures will be exact: whereas Vieta continues this divifion to only one figure in the quotient, and fo obtains but one new figure of the true value of x, or the root of the propofed equation, by every new procefs of his inveftigation; which makes his method of refolving thefe equations exceffively laborious and tedious, when their roots are to be obtained to ten or twelve places of figures. If Vieta had happened to obferve that thefe divifions might be fafely continued to feveral figures in the quotients, or that feveral figures in the quotients, (namely, as many, wanting one, as there are figures in a, or the part of the value of x that is already known,) would be exact, and had confequently directed his readers to continue the

divisions to that number of figures in the quotients, his method would, as I conceive, have been the very same with Mr. Raphson's, and the art of resolving equations of all kinds by approximation would not only have been invented by him, (as it has been,) but would have been brought by him at once to the highest degree of perfection of which it, probably, is capable. He therefore ought to be considered as the original founder of the whole doctrine of resolving equations by approximation, or (as *Lucretius* says of his favourite philosopher *Epicurus*,) as the *Pater et rerum Inventor* on this subject, though Sir Isaac Newton, Mr. Raphson, Monsieur de Lagny, and Dr. Halley, and perhaps some subsequent Mathematicians, have made valuable improvements on his method, by which the practice of it has been very much facilitated.

The End of the Specimen of Vieta's Method of resolving Algebräick Equations.

RE-

REMARKS

ON

THE NUMBER OF *NEGATIVE* AND *IMPOSSIBLE* ROOTS IN ALGEBRÄICK EQUATIONS.

Remarks on the Number of Negative *and* Impoffible *Roots in Algebräick Equations.*

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Article I. SIR Ifaac Newton, Mr. George Campbell, the celebrated Mr. Mac Laurin of Edinburgh, the late Profeffor Waring of Cambridge, and other eminent writers on Algebra have laid down rules for finding the number of impoffible roots in any propofed Algebräick equation; but thefe rules are far from being clear and fatisfactory, and fome of them are abfolutely unintelligible; which indeed is not to be wondered-at, fince they are all founded on a falfe fuppofition, which vitiates all the conclufions derived from it. This fuppofition, (which they lay down as an indifputable, and almoft felf-evident, maxim,) is, "*That every Algebräick equation has as many roots as it has dimenfions,*" though in truth there are very few equations in which this maxim really takes place; to wit, only one fingle form

form in each degree, or order, of equations. Thus, for example, of all the three forms of affected quadratick equations, to wit, $xx + px = q$, $xx - px = q$, and $px - xx = q$, it is only the third form, $px - xx = q$, that can ever have two roots, and *that* only when the absolute term q is less than $\frac{pp}{4}$, or than the square of $\frac{p}{2}$, or of half the co-efficient, p, of the unknown quantity x: and of all the thirteen different forms of affected cubick equations, to wit, $x^3 + px^2 = r$, $x^3 - px^2 = r$, $px^2 - x^3 = r$, $x^3 + qx = r$, $x^3 - qx = r$, $qx - x^3 = r$, $x^3 + px^2 + qx = r$, $x^3 + px^2 - qx = r$, $x^3 - px^2 + qx = r$, $x^3 - px^2 - qx = r$, $-x^3 + px^2 + qx = r$, $-x^3 + px^2 - qx = r$, and $-x^3 - px^2 + qx = r$, it is only the ninth form, $x^3 - px^2 + qx = r$, that can ever have three roots, and *that* only when the absolute term r is less than $\frac{p^3}{27}$, or the cube of $\frac{p}{3}$, or of the third part of, p, the co-efficient of x^2: and of the forty-five different forms of affected biquadratick equations it is only the form $-x^4 + px^3 - qx^2 + rx = s$, or $rx - qx^2 + px^3 - x^4 = s$ that can ever have four roots, and *that* only when the absolute term s is less than $\frac{p^4}{256}$, or the fourth power of $\frac{p}{4}$, or of the fourth part of, p, the co-efficient of x^3. Therefore, when these authors have laid down the foregoing general proposition, (by which they extend what is true in only one form of every new degree, or order, of affected equations, and not

not always even in that one form, to all the other forms of equations of the same order,) they find themselves under a necessity of giving specious names to a parcell of quantities which they endeavour to make pass for roots of these equations, though in truth they are not so, in order to cover the falsehood of their general proposition, and give it, in words at least, an appearance of truth; and with this view they call some of these quantities *negative roots* of the equation to which they relate, and others of them it's *impossible roots*. And to determine the number of the strange quantities so denominated, and discover how many of the supposed roots of a proposed Algebräick equation are *negative*, and how many are *impossible*, has been made by these mysterious writers an object of great importance and most subtle and profound investigation.

Art. 2. That the aforesaid general proposition concerning the number of roots of an equation is false, will appear by examining only one form of an affected cubick equation that has all it's terms compleat; and that form shall be the simplest form that can be chosen, namely, that in which all the terms involving the unknown quantity x are added to each other, and their sum is declared to be equal to a certain known quantity, to wit, the equation $x^3 + px^2 + qx = r$.

Now this equation evidently can have only one root. For, if any particular number, being substituted instead of x in the trinomial quantity $x^3 + px^2 + qx$, would make that quantity, (which is the sum of the three terms

x^3,

x^3, px^2, and qx) become equal to the known quantity r, or, in other words, is the root of this equation, it is evident that any quantity greater, or leſs, than that number, being ſubſtituted in the ſaid trinomial quantity, would make it greater, or leſs, than it was before, and conſequently greater, or leſs, than the abſolute term r, or, in other words, would not be a root of the equation. Here therefore the myſterious Algebräiſts, who maintain that every cubick equation has three roots, find themſelves at a loſs to ſupport their aſſertion by aſſigning any poſſible quantities for the ſecond and third roots of the equation; and therefore they get out of the difficulty by declaring that this equation $x^3 + px^2 + qx = r$ has one real and poſitive, or affirmative, root, and *two impoſſible roots.* This is ſtrange language to be uſed in treating of Algebra, or *Univerſal Arithmetick*, which is, in itſelf, the plaineſt and cleareſt of all ſciences!

Art. 3. In the foregoing equation $x^3 + px^2 + qx = r$ we have ſeen how the myſterious Algebräiſts, in order to ſupport their fundamental, falſe, poſition, are obliged to have recourſe to the introduction of *impoſſible roots.* We will now take another form of cubick equation in which they get rid of their difficulties by the introduction of only *negative roots*, which ſeems to be a fiction ſomewhat leſs bold and abſurd, or, at leaſt, leſs ſhocking to the ear, than that of impoſſible roots. But it is ſtill a fiction, or falſe aſſertion: for it is introducing the roots of another equation, and declaring them to be roots of the equation under conſideration. Let us ſuppoſe the propoſed equation to be $px^2 + qx - x^3 = r$; which,

which, if the abſolute term r is leſs than a certain quantity, may have two roots, but never can have three. Here then theſe writers, to maintain their general poſition, "that every equation has as many roots as it has dimenſions," (according to which this cubick equation ought to have three roots,) find themſelves under a neceſſity of providing it with one root more; which they do in the following manner. They change the ſigns + and — of the terms qx and x^3, (which involve the odd powers of x,) into the contrary ſigns — and +, and thereby convert the original equation $px^2 + qx - x^3 = r$ into the equation $px^2 - qx + x^3 = r$, or $x^3 + px^2 - qx = r$, which is always poſſible, and has only one root; and this one root of this latter equation $x^3 + px^2 - qx = r$ they call a *negative root* of the former equation $px^2 + qx - x^3 = r$. And thus, with this *negative* root of the ſaid original equation $px^2 + qx - x^3 = r$ and it's two former roots, (which really belong to it, if r is leſs than a certain finite quantity,) they compleat the number of three roots which, according to their grand, fundamental, propoſition, it ought to have. And, if r ſhould be greater than the certain finite quantity juſt now alluded-to, and conſequently the original equation $px^2 + qx - x^3 = r$ ſhould become impoſſible, or, rather, falſe *, they then, to maintain their grand, fundamental, propo-

* For an equation is nothing more than a propoſition affirming the equality of one, or more, unknown quantities, or terms involving the powers of an unknown quantity, to a known quantity, and, therefore, when the former quantities cannot be equal to the latter, ought rather to be called *falſe* than *impoſſible.*

ſition,

ſition, (according to which this equation ought to have three roots,) declare that this equation has two impoſſible roots and one negative root.

Art. 4. By the *negative roots* of an equation we are therefore to underſtand the real roots, or, in the language of modern writers of Algebra, the poſitive, or affirmative, roots, of another equation which is derived from the propoſed equation by changing the ſigns + and — that are prefixed to thoſe terms of it which involve in them the odd powers of the unknown quantity x into the contrary ſigns; which *negative* roots of the propoſed equation will often be ſufficient in number to increaſe the number of it's real, or poſitive, or affirmative, roots to the number required by the above mentioned grand, fundamental, propoſition laid down by the myſterious Algebräiſts, to wit, the number of dimenſions of x in the higheſt term of the equation, or the number of units contained in the index of the higheſt power of x. And by *the impoſſible roots* of an equation we are to underſtand certain fictitious quantities which are not the roots of any equation whatſoever, and of which no clear and diſtinct idea can be formed, but which are in number equal to the exceſs of the number of units contained in the index of the higheſt power of x in the equation above the ſum of both the real (or poſitive, or affirmative) roots and the negative roots of the propoſed equation, and are therefore ſufficient to make the whole number of roots of the equation, real, or poſitive, negative, and impoſſible, be equal to the number of the dimenſions of the equation, or to the number of units

in the index of the higheſt power of x in the equation, agreeably to the grand, fundamental, maxim above-mentioned that is ſo much inſiſted-on by modern Algebräiſts.

Art. 5. And hence it is evident that, to find the number of impoſſible roots belonging to any equation, we need only, firſt, aſcertain the number of it's real, or poſitive, roots, and, then that of it's *negative* roots, or of the real, or poſitive, roots of another equation that is derived from the firſt equation by changing the ſigns + and — that are prefixed to all thoſe of it's terms which involve any odd powers of x, and then add theſe two numbers together, and ſubtract their ſum from the number of the dimenſions of the equation, or the number of units in the index of the higheſt power of x in any of it's terms: for the remainder ariſing from this ſubtraction will be the number of the impoſſible roots of the equation.

See on this ſubject the foregoing part of this preſent Collection of Tracts, pages 285, 286, and 287.

THE END.

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